MODELING NON-LINEAR MECHANICAL VIBRATORY PROCESSES USING THE ADOMIAN METHOD: PRELIMINARY RESULTS AND APPLICATIONS IN MECHANICAL ENGINEERING

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Abstract: This research aims to use the Adomian decomposition method for the exact solution of differential equations in modeling mechanical vibratory processes, and for this purpose, a literature review on this method is adopted and promotes the solution of linear and non-linear ordinary differential equations. The method is the Taylor series decomposition of the nonlinear operator of the differential equations in standardized assembly with the generation of a convergent series of the respective operator. Investigations about the method are presented in the form of preliminary research results and were explained, described and applied to the mathematical modeling of a mechanical oscillator, so that the expression to describe the same and then the method was applied to bring the surface your mathematical solutions. In application to the real model, the experimental data were based on virtual tools as well as the entire laboratory environment; however these factors would not interfere with the reliability of the preliminary results that indicate the stability of the analytical solution found.

Keywords: Equation; Differential; Adomian; Operator; Vibration.

MODELAGENS DE PROCESSOS VIBRATÓRIOS MECÂNICOS NÃO-LINEARES COM O USO DO MÉTODO DE ADOMIAN: RESULTADOS PRELIMINARES E APLICAÇÕES NA ENGENHARIA MECÂNICA

Resumo: Esta pesquisa objetiva o uso do método de decomposição de Adomian para a solução exata de equações diferenciais em modelagem de processos vibratórios mecânicos e para tal adota-se uma revisão de literatura sobre este método e promove a solução de equações diferenciais ordinárias lineares e não-lineares. O método trata-se da decomposição em série de Taylor do operador não-linear das equações diferenciais em montagem padronizada com a geração de série convergente do respectivo operador. As investigações acerca do método são expostas sob a forma de resultados preliminares de pesquisa e foram explanadas, descritas e aplicadas à modelagem matemática de um oscilador mecânico, de maneira que a expressão para descrever o mesmo e em seguida aplicou-se o método para trazer a tona suas soluções matemáticas. Em aplicação ao modelo real os dados experimentais foram baseados em ferramentas virtuais bem como todo o ambiente de laboratório, contudo estes fatores não interfeririam na confiabilidade dos resultados preliminares que indicam a estabilidade da solução analítica encontrada.

Palavras-chave: Equação; Diferencial; Adomian; Operador; Vibração.

ISSN: 2357-7592

1. INTRODUCTION

Nowadays, the prediction of behavior and nuances of events in several physical systems has been a necessity demanded by several areas of Engineering, especially Mechanical Engineering, where situations, when physically and mathematically modeled, do not have known analytical solutions, however, in specific situations it is possible, with the proper mathematical treatment, to determine approximate numerical solutions [1, 2, 3, 4].

The presence of nonlinear terms in physical phenomena represents in a more visceral way: an uncertainty. The less we know about what is being studied, the less sure we are of how that component or system will behave for the estimated time interval of its necessary life, and for the reliability of projects it is extremely important to know, for example, when a machine element such as a bolt or a structural element such as a beam will suffer a fracture for having approached the end of its fatigue life, or when a certain machine will need to be calibrated or undergo maintenance again due to the continuous wear of number of cycles of use [3, 4, 5].

In this context, the Adomian Decomposition Method arises, which proposes to solve any and all differential equations analytically, using as a tool some parameters that the author of the method himself created. The Adomian Decomposition Method encompasses a number of advantages, the main ones being its purely analytical nature, which causes its values to be obtained algebraically, however, it is worth noting the convergent nature of the operative, and it can even be exact depending on the equation treated [3, 4].

2. METHODOLOGY

The Adomian decomposition method basically consists in applying differential operators to the equation in question, so that a series expansion of this operator is performed, more precisely a Taylor series, and one of the main postulates of the Adomian Decomposition method [4, 5, 6] considers that the solution can be decomposed as a series of functions, which in mathematical terms leads us to the following general term:

$$y(x) = \sum_{n=0}^{\infty} \left(y_n(x) \right) \tag{2.0}$$

Thus, to describe the method, it is necessary, in the first instance, to explain the characteristics of the linear operator applied by Adomian, starting from the following expression, and the initial design proposed [5,6]:

$$\mathbf{F}\left[y(x) \right] = g(x) \tag{2.1}$$

The following conjecture applies:

$$\mathbf{F} = \mathbf{L} + \mathbf{R} + \mathbf{N} \tag{2.2}$$

The expression denoted above is also a mathematical function and represents an operator of differential nature that is a mixture of the various meanings that an equation can contain, i.e., the linear and nonlinear aparts [6].

Rearranging the initial expression into terms according to the differential operator has:

$$\mathbf{L}\,y(x)+\mathbf{R}\,y(x)+\mathbf{N}\,y(x)=g(x) \qquad (2.3)$$

The effective calculation is recommended to initially isolate the **L** operator, thus obtaining:

$$\mathbf{L}\,y(x)=g(x)-\mathbf{R}\,y(x)-\mathbf{N}\,y(x) \qquad (2.4)$$

In the Adomian Differential Decomposition Method, the differential operator is necessarily of an inverse nature. Thus, applying the knowledge about Linear Algebra operators, the inverse operator is applied to both sides of the previous expression, obtaining:

$$L^{-1}[y(x)]=L^{-1}[g(x)]-L^{-1}[\mathbf{R}\,y(x)]-L^{-1}[\mathbf{N}\,y(x)] \qquad (2.5)$$

Since the applied differential operator admits of an invertible nature, it can be translated as follows, since the operation opposite to derivation is integration:

$$L^{-1}=\int_0^x (F(x)dx) \qquad (2.6)$$

Applying the translation of the differential operator denoted above to the previous expression, generalizing it and performing its operation, we obtain:

$$y(x)=L^{-1}[g(x)]-L^{-1}[\mathbf{R}\,y(x)]-L^{-1}[\mathbf{N}\,y(x)]+C_1 \qquad (2.7)$$

In the expression above, one more term appears in the operation, this term is named is the integration constant C_1, which by definition must appear after solving the operation that names it, moreover, it is related to the initial conditions proposed by the problem [6].

At this stage, Adomian assumes that the nonlinear part of the expression is an analytic function and therefore can and should be written according to the so-called Adomian Polynomials, which have the following structure:

$$\mathbf{N}\,y(x)=\sum_{n=0}^{\infty}(A_n(x)) \qquad (2.8)$$

Keeping in mind the postulates described above, and in Adomian's annals, making the necessary substitutions, we get:

$$\sum_{n=0}^{\infty}(y_n(x))=L^{-1}[g(x)]-L^{-1}[\sum_{n=0}^{\infty}(\mathbf{R}\,y(x))]-L^{-1}[\sum_{n=0}^{\infty}(\mathbf{N}\,y(x))]+C_1 \qquad (2.9)$$

In his postulate Adomian assumes that the solution of the equation can be given by the infinitesimal sum, following a convergence criterion, of its umpteenth given solutions of an inverse differential operator applied to the same expression in its linear parts and in its nonlinear part, is obtained from the sum of the so-called Adomian Polynomials obtained analytically iteratively from its previous terms [6, 7].

The $y(x)$ solutions of the expression, which are the plots of the infinite series shown above, can and should be calculated by comparing the two sides of the equality, thus obtaining the following conjecture for the initial plot:

ISSN: 2357-7592

$$y_0 = L^{-1}[g(x)] + C_1 \qquad (2.10)$$

So, in general, there are:

$$y_{n+1}(x) = -L^{-1}(A_n(x) + \mathbf{R}(y_n(x))) \qquad (2.11)$$

At this point it is worth pointing out, once again, that the other $y(x)$ solutions are obtained iteratively by means of the above equations, also taking into account that the terms after the initial portion depend directly on the Adomian polynomials, as postulated in his original paper [3, 4] and also in his later work [4, 5].

To obtain such polynomials:

$$A_0 = f(u_0) \qquad (2.12)$$

$$A_1 = u_1 \frac{d}{d\mu_0} f(u_0) \qquad (2.13)$$

$$A_2 = u_2 \frac{d}{d\mu_0} f(u_0) + \frac{\mu_1^2}{2!} \frac{d^2}{d\mu_0^2} f(u_0) \qquad (2.14)$$

$$A_3 = u_3 \frac{d}{d\mu_0} f(u_0) + u_1 u_2 \frac{d^2}{d\mu_0^2} f(u_0) + \frac{\mu_1^3}{3!} \frac{d^3}{d\mu_0^3} f(u_0) \qquad (2.15)$$

$$\ldots$$

However, for the initial case, the expression for Adomian Polynomials and according to Ronni's translation [7] the equation can also be expressed as follows:

$$A_n(x) = \frac{1}{n!} \cdot \frac{d^n}{d\lambda^n} [\mathbf{N} \cdot \sum_{i=0}^{n} (y_i(x)\lambda^i)]_{\lambda=0} \qquad (2.16)$$

From this point on, you arrive at a structure such that you have the following algorithm: you find the initial portion y_0; from y_0 is found A_0; from A_0 is found y_1; from y_1 is found A_1; from A_1 is found y_2; and this process continues until the expression begins to converge, or the null value, in the latter case, would indicate that both the equation and the method become exact and no longer converge [5, 6, 7, 8].

In this case, the initial Adomian polynomial is:

$$A_0(x) = \mathbf{N}\, y_0(x) \qquad (2.17)$$

And the other Adomian polynomials would be as follows:

$$A_1(x) = y_1 \mathbf{N}\, y_0(x) \qquad (2.18)$$

$$A_2(x) = \frac{y_1^2(x)}{2} \cdot \mathbf{N''}\; y_0(x) + y_2 \mathbf{N'}\, y_0(x) \qquad (2.19)$$

$$A_3(x) = y_3 \mathbf{N'}\, y_0(x) + y_1(x) y_2 \mathbf{N''}\; y_0(x) + \frac{y_1^2(x)^3}{3!} \cdot \mathbf{N'''}\, y_0(x) \qquad (2.20)$$

$$A_4 = y_4 \mathbf{N'}\, y_0(x) + \left(y_1(x) y_3(x) + \frac{y_1^2(x)^3}{3!} \right) \mathbf{N''}\, y_0(x) + \left(\frac{y_1(x)^2 y_2(x)}{2} \right) \mathbf{N'''}\, y_0(x) \qquad (2.21)$$

...

And consecutive to the generating formula to obtain the final value $y(x)$ of the solution of the equation, it remains to sum the values obtained from the solutions $y_n(x)$ arranged in the form of a Taylor series, as predicted by the postulate.

$$y(x) = \sum_{n=0}^{\infty} (y_n) = y_0 + y_1 + y_2 + y_3 + y_4 + ... + y_n \qquad (2.22)$$

Finally, the result of the expression converges to $y(x)$ whenever there is a α value greater than or equal to 0 and less than 1.

Moreover, it must belong to the group of naturals, satisfying the following relation: $\|y_{n+1}\|_\infty \leq \alpha \|y_n\|_\infty$ for every $k \geq n_0$.

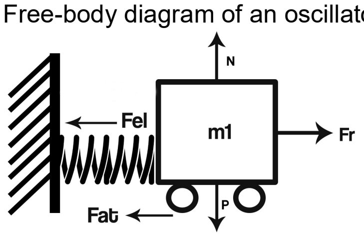
3. RESULTS AND DISCUSSION

3.1. Characteristics of vibratory processes

In life we come across numerous physical phenomena associated with mechanical vibrations and their manifestations. Vibration is a physical phenomenon and is generally associated with the dissipation of kinetic energy and its eventual conversion from one type of energy to another for various reasons; among the most common are discontinuities of the materials involved in the system. In other words, vibration is an indicator of problems or discontinuities in systematic mechanical processes [8, 9, 10, 11].

To explain the method in practical terms, consider the following schematic diagram:

Image #01: Free-body diagram of an oscillatory system



(Source: Author's own; 2021)

The system consists of a cart of mass m_1 on which a F_r resultant force is applied to the abscissa axis, designating its trajectory. Here there is also an elastic force F_{el} from the spring, given in the opposite direction to the motion of the cart, also the frictional force F_{at} occurs between the wheels of the vehicle and the firmament [11, 12].

Finally, we highlight the forces present in the vertical direction, which in this case are the contact forces, the P weight, and N the normal action force. It is worth noting that the mass m1 is treated here as the mass of the assembly as a whole.

3.2. Normalization to the Adomian Differential Decomposition Method

Following the algorithm mentioned above during the description of the method, the first step is based on deciphering the terms of the equation, classifying them as its linear, nonlinear and auxiliary function, if any.

The following equation will then be considered for the execution of the method:

$$\frac{d^2y}{dt^2} + \omega_0^2 \cdot y^2 t^3 + \mu \cdot g = 16t \qquad (3.1)$$

Since the physical situation considered refers to a mass spring oscillator system, there are many variables that influence the behavior of the spring and consequently will influence the modeling of the system, analogously, we can associate the expression to the behavior of a nonlinear spring in the system [13, 14, 15].

Then, the following operative procedure is followed, in the expression, the term:

$$\mathbf{L} = \frac{d^2y}{dt^2}, \qquad (3.2)$$

In this case, it is the highest order derivative term.

The linear **R** term is given by:

$$\mathbf{R} = \omega_0^2 \cdot y^2 + \mu \cdot g \qquad (3.3)$$

The nonlinear **N** term:

$$\mathbf{N} = t^3 \qquad (3.4)$$

Finally, the auxiliary function of independent variable $g(x)$ is given by:

$$g(x) = 16t \qquad (3.5)$$

Then the Adomian inverse operator is applied to both sides of the equation, thus obtaining:

$$L^{-1}\left[\frac{d^2y}{dt^2} + \omega_0^2 \cdot y^2 t^3 + \mu \cdot g\right] = L^{-1}[16t] \qquad (3.6)$$

It is worth noting that for the execution of the equation there are the following boundary conditions:

$$y(0) = y_0 \qquad (3.7)$$

$$t = 0 \qquad \frac{dy}{dt} = 0 \qquad (3.8)$$

Decomposing the operation into all the terms of the expression, we arrive at:

$$y(t) - y(0) - \frac{dy(0)}{dt}t + L^{-1}[\omega_0^2 \cdot y^2 t^3] - \mu \cdot g \cdot y \cdot t = -8t^2 y + \varphi \qquad (3.9)$$

Here also φ another constant of the integration process arises.

Applying the boundary conditions proposed by the problem, it is possible to obtain:

$$y(t) + \omega_0^2 \cdot y^2 \cdot L^{-1}[t^3] - \mu \cdot g \cdot y \cdot t = -8t^2 y + \varphi \qquad (3.10)$$

Thus, applying the assumption in 2.0. Equation, we have the following translation:

ISSN: 2357-7592

$$\sum_{n-0}^{\infty}(y_n(t)) = -8t^2 y - \omega_0^2 \cdot y^2 \cdot L^{-1}[t^3] + \mu \cdot g \cdot y \cdot t + \varphi \qquad (3.11)$$

In addition to applying the above assumption, one must assume that the nonlinear part of the equation can be written in terms of Adomian polynomials, in this particular case, the following expression is obtained:

$$\mathbf{N}\, y(x) = t^3 = \sum_{n=0}^{\infty}(A_n(x)) \qquad (3.12)$$

In this case, for Adomian polynomials the independent variable is, representing the system and the non-permanent regime.

Applying it to the expression, you get:

$$\sum_{n-0}^{\infty}(y_n(t)) = -8t^2 y - \omega_0^2 \cdot y^2 \cdot L^{-1}[\sum_{n=0}^{\infty}(A_n(t))] + \mu \cdot g \cdot y \cdot t + \varphi \qquad (3.13)$$

After performing the comparison process with the two ends of the equation, the following recurrence relation is verified:

$$y_0(t) = \gamma_0 \qquad (3.14)$$

Generalizing the solution term of the equation, we have:

$$y_{n-1}(t) = -\omega_0^2 \cdot y^2 \cdot L^{-1}[A_n(t)] + \mu \cdot g \cdot y \cdot t + \varphi \qquad (3.15)$$

Where the A_0 initial Adomian polynomial is given by:

$$A_0(t) = \mathbf{N}\, y(t) = t^3 \qquad (3.16)$$

We will obtain:

$$A_0(t) = \gamma_0^3 \qquad (3.17)$$

Thus, you have:

$$y_1(t) = \omega_0^2 \cdot y^3 \cdot \frac{t^4}{4} + \mu \cdot g \cdot y \cdot t + \varphi \qquad (3.18)$$

Similarly, we have the posterior Adomian polynomial given by:

$$A_1(t) = y_1(\mathbf{N'}\, y(t))_{y=y_0} = \omega_0^2 \cdot y^3 \cdot \frac{3t^6}{4} \qquad (3.19)$$

Bringing as a result:

$$y_2(t) = -\omega_0^4 \cdot \frac{3t^7 \cdot y^7}{112} + \mu \cdot g \cdot y \cdot t + \varphi \qquad (3.20)$$

The next solution term, following 2.11. Equation has value:

ISSN: 2357-7592

$$y_3(t) = \frac{17\omega_0^8 y^{22} - 48t^{17}\omega_0^8 y^4}{30464} + \mu \cdot g \cdot y \cdot t + \varphi \tag{3.21}$$

In turn, using the term obtained previously, one obtains:

$$A_3(x) = \left[\frac{17\omega_0^8 y^{22} - 48t^{17}\omega_0^8 y^4}{30464} \cdot 3t^2 \cdot \gamma_0\right] + \left[\omega_0^2 \cdot y^3 \cdot \frac{t^4}{4} \cdot -\omega_0^4 \cdot \frac{3t^7 \cdot y^7}{112} \cdot \gamma_0\right] + \left[\frac{\left(\omega_0^2 \cdot y^3 \cdot \frac{t^4}{4}\right)^2 t^3}{3!} \cdot 6 \cdot \gamma_0\right] \tag{3.22}$$

Solving can be done up to this point and the results will be satisfactory, that is, up to the A_3 term, however, the more iterations that are done the better the convergence of the method, and can raise the solution brought by the equation to accuracy.

However, if we go back to the proposed statement in the situation, the spring employed is not linear, assuming the use of a conical helical spring, for example, its oscillator term then depends on other material quantities such as the shear modulus (G), specific to the spring material, the nominal diameter of the coil (D_0) the number of active coils (n_a), and the torsional moment of inertia of the spring (J). Thus, performing the appropriate substitution:

$$\omega_0 = \frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a} \tag{3.23}$$

Therefore, the final solution value of the modeled and idealized equation for the situation using the Adomian Differential Method is given by:

$$y(x) = \sum_{n=0}^{\infty}(y_n) = \gamma_0 + (\mu \cdot g \cdot y \cdot t + \varphi)$$
$$\left(\begin{array}{l} \left(\frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a}\right)^2 \cdot y^3 \cdot \frac{t^4}{4} - \left(\frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a}\right)^4 \cdot \frac{3t^7 \cdot y^7}{112} + \frac{17\left(\frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a}\right)^8 y^{22} - 48t^{17}\left(\frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a}\right)^8 y^4}{30464} + \\ \left[102\gamma_0^3\left(\frac{4G \cdot J \cdot m_1}{\pi \cdot D_0^3 \cdot n_a}\right)^{26} y^2\left[\left(\frac{9t^{20}y^5}{190400} - \frac{t^3 y^{23}}{700672}\right)\right] + \left[\frac{-t^{12}y^{11}}{19712}\right] + \left[\frac{-t^{12}y^7}{8064}\right]\right] + \ldots + y_n \end{array}\right) \tag{3.24}$$

4. CONCLUSION

With the above, the Adomian decomposition method was presented in application to mechanical vibratory systems in order to prove capable of presenting exact solutions for them, including for nonlinear modeling, resulting effective for day-to-day applications in Mechanical Engineering and especially for vibratory systems in general.

These results are preliminary in the current research proposal, however, we already have a mechanical oscillatory system, linear and nonlinear, with an exact solution for theoretical values, leaving as next actions the proposal to use the associated database for testing with real values.

Acknowledgments

The authors thank the Centro Universitário SENAI Cimatec and the Universidade Estadual da Bahia (UNEB) for the incentive, support and logistics associated.

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ISSN: 2357-7592