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DYNAMICS OF STEPPED EULER-BERNOULLI BEAMS WITH ELASTIC END SUPPORTS

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Abstract. *Stepped beams with elastic end supports have been extensively investigated due to their importance in structural engineering fields, including active structures, structural elements with integrated piezoelectric materials, shaft-disc system components, turbomachinery blades, etc. In the present work, a mathematical modeling is proposed for stepped beams with elastic end supports. The analysis is based on the classical Euler-Bernoulli beam theory. In comparison with the published literature on the transverse vibration of single cross-section change beams, there are relatively few works covering beam vibration when there is more than one change in the beam cross-section. In the present study, the natural frequencies and the mode shapes of beams with two step changes in cross-sections are investigated. Combinations of the classical clamped, pinned, sliding and free type end supports are considered. The first three natural frequencies of the studied beams are evaluated for some types of end supports. The proposed method can be extended to beams with any number of changes in cross-section.*

Keywords: *stepped beam, natural frequency, mode shape.*

1. INTRODUCTION

A brief review of selected publications on transverse vibration of beams with changes in cross-sections follows. Taleb and Suppiger (1961) and Levinson (1976) derived the frequency equation for a simply supported stepped beam. Heidebrecht (1967) showed

numerical method to calculate the first natural frequency of simply-supported beams. Jang and Bert (1989a) and Jang and Bert (1989b) were the first to derive the frequency equations as fourth order determinants equated to zero, for combinations of the classical clamped, pinned and free end supports. Vibration analysis of stepped beam with one step cross-section change subject to the constraining effect of rotational and translational springs at both ends was presented by Maurizi and Bellés (1993). De Rosa (1994) studied the vibration of a beam with one step change in cross-section with elastic supports at the ends. Neguleswaran (2002), studied the frequency equations of an Euler-Bernoulli beam with up to three step changes in cross-section and on classical and/or elastic supports. The first three natural frequencies for three different types of transverse sections of beams, was tabulated by this author.

Dong (2005) presented a scheme to calculate the laminated composite beam's flexural rigidity and transverse shearing rigidity based on first order shear deformation theory. A stepped beam model was then developed by using Timoshenko's beam theory to predict analytically the natural frequencies and mode shapes of a stepped laminated composite beam. Modal analysis with piezoelectric materials bonded on beam surface, i. e., stepped piezoelectric beams, was validated by Maurini et al (2006). They used Euler-Bernoulli model from finite element analysis and experimental procedures validated the results.

In Stanton and Mann (2010) was developed an analytic framework for determining closed form expressions for the natural frequencies, mode shapes, and frequency response function for Euler–Bernoulli beams with any number of geometric or material discontinuities. Theoretical predictions are experimentally validated as well.

The present paper presents the transverse vibration of Euler-Bernoulli beams with discontinuous geometry and elastic end supports. The natural frequencies and the mode shapes of stepped beams are discussed and compared to each other. Combinations of the classical clamped, pinned, sliding, and free types of elastic end supports are considered. The first three frequencies parameters of beams with two step changes in cross-section are evaluated for selected sets of system parameters and types of end supports. The proposed method can be extended to beams with any number of step changes in cross-section.

2. MATHEMATICAL FORMULATION

According to Euler-Bernoulli beam's theory, the equation of a clamped-free uniform beam in transversal vibration is obtained by applying the static equilibrium equations to sum the forces and moments that act in the beam. The differential equation of the transverse free vibration of a slender beam is as follows, Inman (2001).

$$\frac{\partial^2 v(x,t)}{\partial t^2} + c^2 \frac{\partial^4 v(x,t)}{\partial x^4} = 0\,. \tag{1}$$

where $c = \sqrt{EI/\rho A}$, EI is the flexural rigidity (E is Young's modulus for the beam (N/m^2) and I is the cross-sectional area moment of inertia (m^4)), ρ is the mass density (kg/m^3), A is the cross-section area (m^2), $v(x,t)$ is the deflection of the beam (m), x is the spatial abscissa (m) and t is the time (s).

The solution of Eq. (1) subject to four boundary conditions and two initial conditions are used to obtain system of linear equations in order to determine the constants of general solution. The Eq. (1) is simplified by assuming a separation of variables solution of the form as follows:

$$v(x,t) = X(x)\mathrm{T}(t) \tag{2}$$

When one substitutes the Eq. (2) into the Eq. (1), the equation of motion turns:

$$c^2 \frac{X^{iv}(x)}{X(x)} = -\frac{\ddot{\mathrm{T}}(t)}{\mathrm{T}(t)} = \omega^2\,. \tag{3}$$

The Eq. (3) can be rearranged as:

$$X^{iv}(x) - \left(\frac{\omega}{c}\right)^2 X(x) = 0 \,. \tag{4}$$

We define:

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \,. \tag{5}$$

β is the dimensional natural frequency (m); ω is the angular natural frequency (rad/s).

The general solution of Eq. (4) can be put in the form, Inman (2001).

$$X(x) = B_1 \sin \beta x + B_2 \cos \beta x + B_3 \sinh \beta x + B_4 \cosh \beta x \qquad 0 \le x \le L \tag{6}$$

where $X(x)$ represents the mode shape of beam, B_1, B_2, B_3 and B_4 are coefficients of general solution and L is the length of continuous beam.

Based on the Euler-Bernoulli beam, one can study stepped beams with several step changes in cross-section and with different elastic supports as shown in Fig. 1 where (n is the number of sections of the beam).

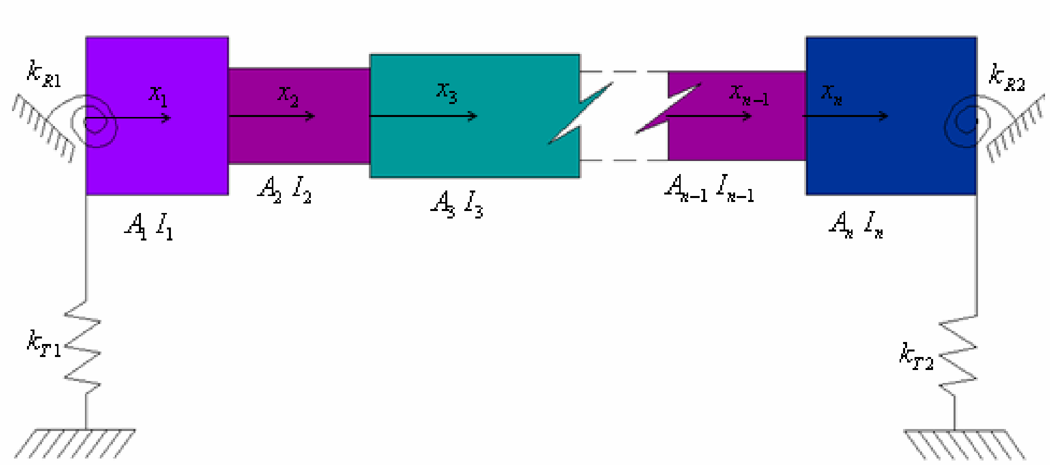

Figure 1. Stepped beam with multiple step changes in cross-section, Vaz (2009).

where $0 \le x_i \le L_i$, $i = 1, 2, \cdots, n$, L_i is the length of the segment of the beam (m), k_{R1}, k_{R2} and k_{T1} are the rotational and translation spring constants, A_n is the cross-section area of n^{th} segment (m^2) and I_n is the cross-sectional area moment of inertia (m^4).

The general solution for each section of the stepped beam in multiple steps.

$$X_{i,k}(x_i) = B_{b_j}\, sen\beta_{i,k} x_i + B_{b_j} \cos \beta_{i,k} x_i + B_{b_j} senh\beta_{i,k} x_i + B_{b_j} \cosh \beta_{i,k} x_i \qquad 0 \le x_i \le L_i \tag{7}$$

where $i = 1, \ldots, n$ is the section number of the beam, k is the number of mode shape, b_j (where $j = 1, \ldots, 4$) is the index of the coefficient of the general solution.

The index of the coefficient of i^{th} section of the beam can be expressed as follows.

$$b_j = j + 4(i-1) \tag{8}$$

where $j = 1, \dots, 4$ is the number of coefficient of the general solution.

2.1. Boundary conditions

The vibration equation, Eq. (7), contains four unknown coefficients, B_{b_j}, and one natural frequency for each segment of beam. Hence, the calculating of solution of Eq. (7) requires four boundary conditions for the end and others four boundary conditions for each one of the junction of the different segments of beam.

The boundary conditions are obtained by examining the deflection, the slope, the bending moment and the shear force at each end of the beam. In addition to satisfying four boundary conditions, the solution of Eq. (1) can be calculated only if two initial conditions (in time) are specified.

The eigenvalue problem must be solved for a particular set of boundary conditions, resulting in expressions for the eigenfunctions $X_{i,k}(x_i)$ and frequencies ω which the structure can accommodate in free vibration. The boundary conditions for the structural system under consideration, Fig. 1, are as follows.

In the ends:

at $x = 0$

- Bending moment

$$EI_1 \left. \frac{\mathrm{d}^2 X_1(x_1)}{\mathrm{d}x_1^{\,2}} \right|_{x_1=0} = k_{R1} \left. \frac{\mathrm{d}X_1(x_1)}{\mathrm{d}x_1} \right|_{x_1=0} . \tag{9}$$

- Shear force

$$EI_1 \left. \frac{\mathrm{d}^3 X_1(x_1)}{\mathrm{d}x_1^3} \right|_{x_1=0} = -k_{T1} \left. X_1(x_1) \right|_{x_1=0} . \tag{10}$$

at $x_n = L_n$

- Bending moment

$$EI_n \left. \frac{d^2 X_n(x_n)}{dx_n^{\,2}} \right|_{x_n=L_n} = k_{R2} \left. \frac{dX_n(x_n)}{dx_n} \right|_{x_n=L_n} . \tag{11}$$

- Shear force

$$EI_n \left. \frac{d^3 X_n(x_n)}{d{x_n}^3} \right|_{x_n = L_n} = -k_{T2} \left. \frac{dX_n(x_n)}{dx_n} \right|_{x_n = L_n} . \tag{12}$$

The continuity conditions at the junctions are:

- Deflection

$$X_{p-1}(x_{p-1})\Big|_{x_{p-1}=L_{p-1}} = X_p(x_p)\Big|_{x_p=0} \quad , \quad p = 2, \cdots, n . \tag{13}$$

- Slope

$$\left. \frac{\mathrm{d}X_{p-1}(x_{p-1})}{\mathrm{d}x_{p-1}} \right|_{x_{p-1}=L_{p-1}} = -\left. \frac{\mathrm{d}X_p(x_p)}{\mathrm{d}x_p} \right|_{x_p=0} . \tag{14}$$

- Bending moment

$$I_{p-1} \left. \frac{\mathrm{d}^2 X_{p-1}(x_{p-1})}{\mathrm{d}x_{p-1}^2} \right|_{x_{p-1}=L_{p-1}} = I_p \left. \frac{\mathrm{d}^2 X_p(x_p)}{\mathrm{d}x_p^2} \right|_{x_p=0} . \tag{15}$$

- Shear force

$$I_{p-1} \left. \frac{\mathrm{d}^3 X_{p-1}(x_{p-1})}{\mathrm{d}x_{p-1}^3} \right|_{x_{p-1}=L_{p-1}} = -I_p \left. \frac{\mathrm{d}^3 X_p(x_p)}{\mathrm{d}x_p^3} \right|_{x_p=0} . \tag{16}$$

These boundary conditions are used to obtain the general solution and a system of homogeneous algebraic equations in the unknowns B_{bj} . In order to have a non-trivial solution, the determinant of the coefficient matrix must vanish identically.

3. NUMERICAL RESULTS

The results for two different stepped beams are presented in this section, one of the beams presents a single step change in cross-section. The other beam has two step changes. Both beams are supported on elastic ends. Numerical results for the first three natural frequencies for different end support were compared to available literature.

Table 1 and Tab. 2 lists the dimensionless natural frequencies, $\hat{\beta}_{1,1}$, of stepped beam with one step change in cross-section, with elastic end supports and ratio of moments of inertia of adjacent segments. The goal of this test is to verify the behavior of beam when half of the beam has its cross section elongated or shortened. The calculations were carried out assuming a stepped beam with lengths equal to $L_1 = L_2 = L/2$ and ratios moments of inertia, as presented in Eq. (17), starting in $\overline{I}_1 = 0.1$ and finishing in $\overline{I}_1 = 10$. It is possible to note that when $\overline{I}_1 = 1$ the both beam cross-sections are equal in size and in this case the beam is continuous.

$$\overline{I}_1 = \frac{I_2}{I_1} \tag{17}$$

where $\overline{I}_1$ is the ratio between adjacent moments of inertial, I_1 (m^4) and I_2 (m^4).

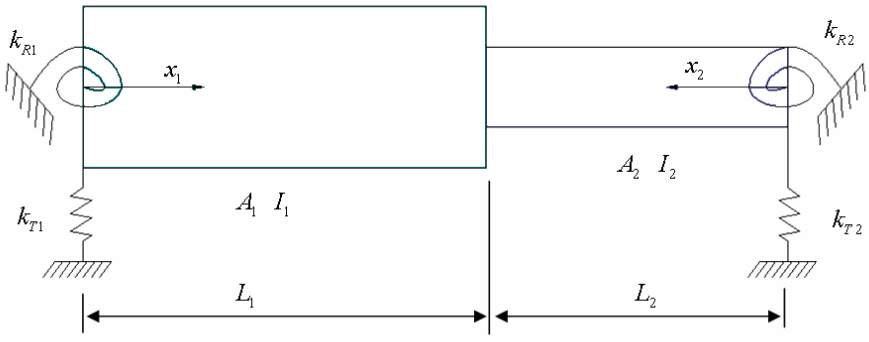

Figure 2. Beam with one step change in cross-section, Vaz (2009).

Table 1. First dimensionless natural frequencies of a single stepped beam with elastic support in one of the ends and free in other end.

Supports	$R_1 = T_1$	$R_2 = T_2$	$\hat{\beta}_{1,1}$		
			$\overline{I}_1 = 0.1$	$\overline{I}_1 = 1$	$\overline{I}_1 = 10$
Free-free	∞	∞	0	0	0
—	500	∞	0.34821	0.29263	0.22976
—	5	∞	1.09088	0.91389	0.71583
—	0.05	∞	2.17505	1.81072	1.3883
Clamped-free	0	∞	2.2355	1.8751	1.43628

where R_1, T_1, R_2 and T_2 are rotational and translational dimensionless parameters. $\hat{\beta}_{1,k}$ is the dimensionless natural frequency; the index 1 represents the first segment beam for k^{th} natural frequency. The rotational and translational dimensionless parameters, the dimensionless natural frequencies $\hat{\beta}_{1,k}$ and the angular natural frequencies, ω_n, are expressed as follows.

$$R_1 = \frac{EI_1}{k_{R1}L_1},\quad T_1 = \frac{EI_1}{k_{T1}L_1^3},\quad R_2 = \frac{EI_2}{k_{R2}L_2},\quad T_2 = \frac{EI_2}{k_{T2}L_2^3}. \tag{18}$$

$$\hat{\beta}_{1,k} = \beta_{1,k} L. \tag{19}$$

$$\omega_n = \left(\frac{\hat{\beta}_{1,k}}{L}\right)^2 \sqrt{\frac{EI_i}{\rho A_i}}. \tag{20}$$

Table 2. First dimensionless natural frequencies of a stepped beam clamped (R1=T1 = 0) in one of the ends and with elastic support in other end.

Supports	$R_1 = T_1$	$R_2 = T_2$	$\hat{\beta}_{1,1}$				
			$\bar{I}_1 = 0.1$	$\bar{I}_1 = 0.5$	$\bar{I}_1 = 1$	$\bar{I}_1 = 5$	$\bar{I}_1 = 10$
Clamped-free	0	∞	2.2355	2.00987	1.8751	1.56119	1.43628
	0	500	2.23663	2.01208	1.87866	1.57413	1.45941
	0	50	2.24656	2.03147	1.90954	1.67694	1.62735
	0	5	2.33168	2.1873	2.13952	2.20142	2.29838
	0	0.5	2.70056	2.77289	2.87787	3.24695	3.40801
	0	0.05	3.42543	3.83194	4.0691	4.56259	4.74954
	0	0.005	3.88099	4.42004	4.65386	5.03687	5.20451
Clamped-clamped	0	0	3.94537	4.50112	4.73004	5.095	5.26124

The Figures (3) and (4) present the mode shapes of continuous and stepped beam, respectively, with the same conditions of elastic supports. The boundary conditions have been modified by using different translational parameters and keeping the rotational parameters ($R_1 = R_2 = \infty$).

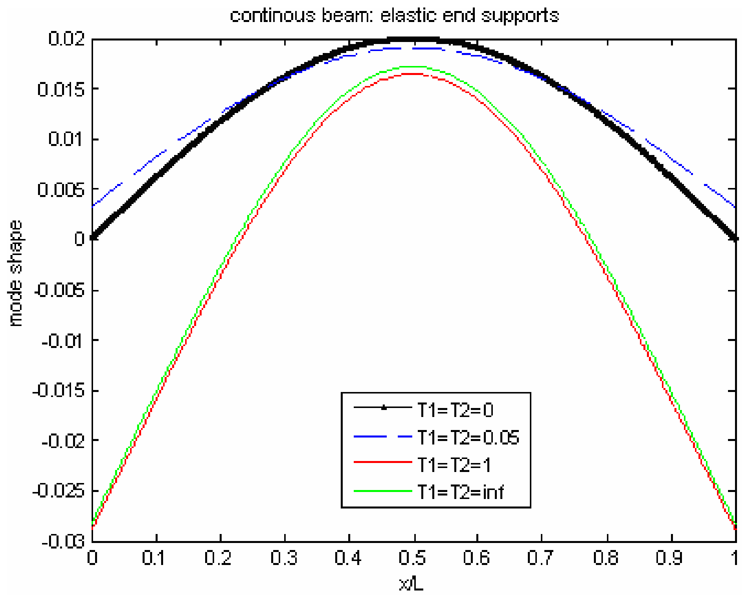

Figure 3. First mode shape of continuous beam with dimensionless parameters of translational.

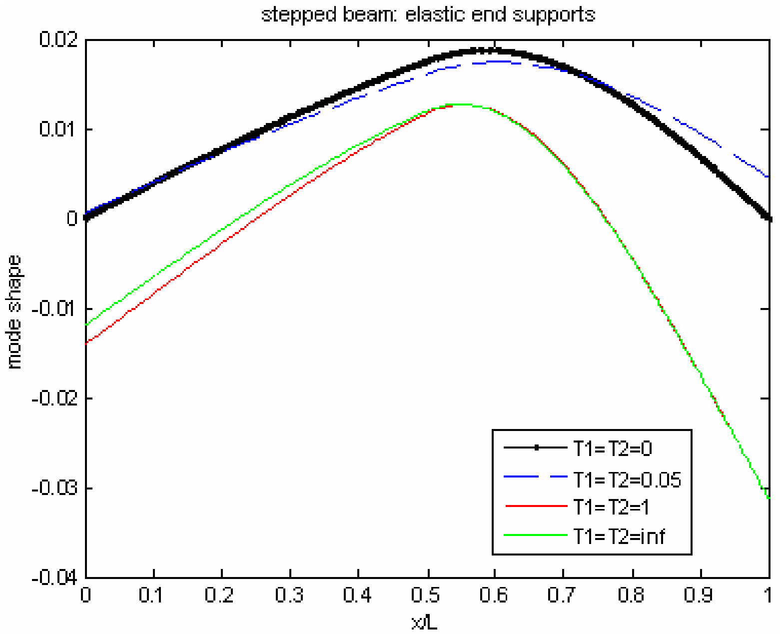

Figure 4. First mode shape of stepped beam with dimensionless parameters of translational.

The first three dimensionless beam's natural frequencies with two step changes in cross-sections are showed in Tab. 3 to Tab. 5. The beam lengths are $L1$ = 0.200 (m), $L2$ = 0.300 (m), and $L3$ = 0.500 (m). The main dimensions related to cross-section depends on beam type, that is, type 1,and type 2 are rectangular cross-section beam, and type 3 is circular cross-section beam, as follows:

- for type 1: rectangular cross-section with constant height and the following values for width, $b1$ = 0.005 (m), $b2$ = 0.006 (m), and $b3$ = 0.009 (m)
- for type 2: rectangular cross-section width constant width and the following values for height, $h1$ = 0.005 (m), $h2$ = 0.006 (m), and $h3$ = 0.009 (m)
- for type 3: : circular cross-section with the following values for the diameters, $d1$ = 0.005 (m), $d2$ = 0.006 (m), and $d3$ = 0.009 (m)

Table 3. First three dimensionless frequencies of a stepped beam with three cross-sections - type 1.

Classical end supports	R1	T1	R2	T2	Type 1		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
clamped-free	0	0	∞	∞	1.66100	4.56222	7.84841
free-free	∞	∞	∞	∞	0	4.77621	7.91134
clamped-sliding	0	0	0	∞	2.20800	5.42355	8.62546
clamped-pinned	0	0	∞	0	3.80416	7.04505	10.1739

Table 4. First three dimensionless frequencies of a stepped beam with three cross-sections - type 2.

Classical end supports	R1	T1	R2	T2	Type 2		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
clamped-free	0	0	∞	∞	1.71452	5.16922	9.41405
free-free	∞	∞	∞	∞	0	5.57601	9.51969
clamped-sliding	0	0	0	∞	2.57846	6.32019	10.2630
clamped-pinned	0	0	∞	0	4.39742	8.43703	11.8825

Table5. First three dimensionless frequencies of a stepped beam with three cross-sections - type 3.

Classical end supports	R1	T1	R2	T2	Type 3		
					$\hat{\beta}_{1,1}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$
clamped-free	0	0	∞	∞	1.50383	4.93207	9.42708
free-free	∞	∞	∞	∞	0	5.54988	9.59866
clamped-sliding	0	0	0	∞	2.42957	6.28097	10.1805
clamped-pinned	0	0	∞	0	4.18529	8.50978	11.7550

The mode shapes of stepped beams with two step changes in cross-section are shown in Figures 5(a) to 5(c). The different colors indicated in Figures 5(a), 5(b) and 5(c) represent the different segments of discontinuous beam.

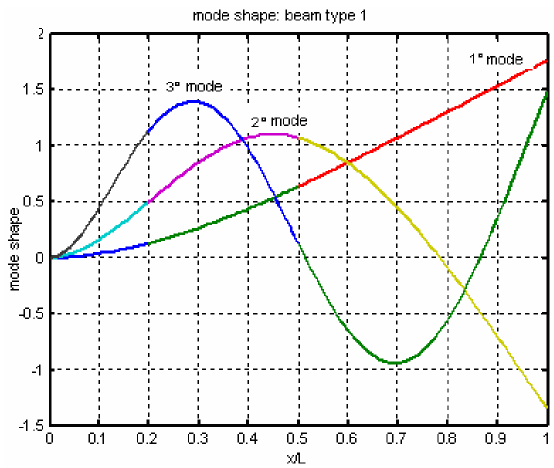

Figure 5 (a).

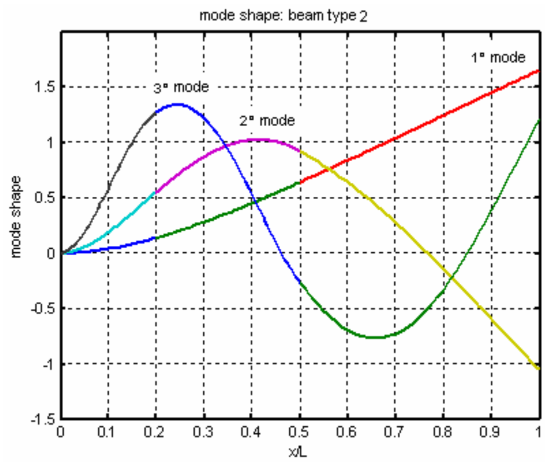

Figure 5(b).

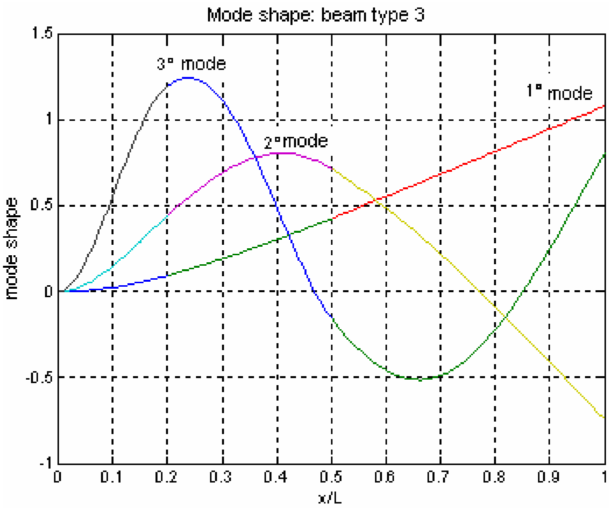

Figure 5(c).

Figure 5. The first three mode shapes for three different types of cross-section: (a) the cross-section is rectangular with constant height; (b) the cross-section is rectangular with constant width; (c) the cross-section is circular.

4. SUMMARY AND CONCLUSIONS

This work presents Euler-Bernoulli Beam theory known as elementary theory in order to evaluate the natural frequencies and the mode shapes of stepped beams in multiple parts. The characteristic equation is function of the dimension of the individual parts and the general constraints at the end of the parts. The general solution, Eq. (4), gives the dimensionless frequencies, $\beta_{i,k}$, for the beam and allows to study the several parameters obtained from these frequencies, as the dimensionless natural frequencies $\hat{\beta}_{1,k}$ and the angular natural frequencies, ω_n. The coefficient matrix is formulated by applying the boundary conditions into Eq. (4). The natural frequencies of the transverse vibrations of a stepped beam are obtained by setting the determinant of the coefficient matrix to vanish and then the mode shapes can be calculated. To clarify the proposed method, numerical simulations have been presented for a beam with one step change in cross-section and two step changes in cross-section for different elastic end supports. Three types of cross-section area were considered as the rectangular with constant height, the rectangular cross-section with constant width and the circular. The numerical results from discontinuous geometry beam model confirm the validity of the approach and hence, exact methods such as the proposed beam are required for practical implementation of such discontinuous structures.

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