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DETERMINATION OF BUCKLING LOAD OF STRUCTURES WITH THE PRESENCE OF BODY FORCE

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Abstract. *The study is aimed at developing a numerical algorithm for solving a constrained eigenvalue problem (CEVP). The CEVP sees its engineering applications in the determination of the buckling load of structures when the body force plays a critical role. As opposed to some CEVP in which some eigenvectors are coerced to a specific manner, the mathematical statement of the current CEVP is completed with a generalized eigenvalue problem containing an unknown parameter. An equality constraint on one of the eigenvalues is identified. The algorithm presented is a method of trial and error with which an interval embracing the target eigenvalue is determined. Through linear interpolation, the eigenvalue satisfying the constraint is calculated. The algorithm is applied to several engineering problems including determination of the buckling loads of a 2D crane structure under gravitational effect, and the critical payload a plane frame can hoist under the influence of gravitational acceleration. The accuracy is demonstrated using an example whose classical solution exists. The significance of the equality constraint in the EVP is shown by comparing the solutions without the constraint on the eigenvalue. Effectiveness and accuracy of the numerical algorithm are presented.*

Keywords: buckling, constraint, eigenvalue, gravity

1. INTRODUCTION

Buckling has been one of the main concerns in structure design against catastrophic failure for a long time. Naturally the topic has attracted a large group of researchers and engineers in the past rendering a rich source of articles in the area. A few textbooks in theoretical settings as well as numerical practices have been published and used in academia, such as [1]~[3], which provide a good source of references in the related fields. Further, the numerical procedures of finding the buckling loads have been implemented in a few commercial codes for engineering practices, for example, [4] ~ [7]. The study regarding the buckling of the elastic object subject to gravity as well as other applied loads has received much less attention. Roberts and associates [8], [9] investigate the lateral buckling of an elastic I-beam subject to uniformly distributed load using energy method. Influence of such parameters as sectional warping rigidity, location of applied load with respect to the shear center is thoroughly studied. Dougherty [10], [11] considers the lateral buckling of an elastic beam subject to uniformly distributed load as well as a central point load and end moments. In the studies, gravity load of the beam is modeled as a uniformly distributed load applied on the top surface of the beam. A numerical approach is employed to solve for the critical load for the beam.

The loads applied on the beam in the study [10], [11] appear to be proportional in that the point force and the uniformly distributed load, for example, vary at the same rate, if necessary. In this currently study, gravitational load and other applied forces are non-proportional. Thus the buckling problem under the influence of gravity is formulated as a constrained eigenvalue problem. Kerstens [12] provides a review of methods employed in solving constrained eigenvalue problems. Cheng et al. [13] present a classic study of the buckling of a thin circular plate. In the study, Ritz method is employed to solve the first buckling load of the circular plate with boundary fixed. The only load is the in-plane gravity. Kumar and Healey [14] present a study of stability of elastic rods. The generalized eigenvalue problem consists of a set of constraint equations imposed on the nodal displacements of the model. There is no constraint on the eigenvalue itself. Efficient numerical methods are presented to solve the first few lowest natural eigenvalues. Zhou [15] examines an algorithm for the design optimization of structure systems subject to both displacement as well as eigenvalue (natural frequency) constraints. An iterative algorithm based on Rayleigh Quotient approximation is shown to be efficient in solving the dual constraint eigenvalue problems.

In this paper, the problem to be tackled is given and formulated in mathematical form in Section 2. The deviation of the current problem from the others is disclosed. It is shown that addressing the current problem using the usual treatment would lead significant errors. Section 3 presents a simple algorithm for solving the problem efficiently. The proposed algorithm is tested using three numerical examples in Section 4. It is seen from the examples that the proposed algorithm has achieved excellent accuracy.

2. MATHEMATICAL STATEMENT OF THE CURRENT PROBLEM

To determine the buckling load of a structure, the following eigenvalue problem needs to be solved.

$$[\mathbf{K} - \lambda \mathbf{K}_f]\mathbf{U} = \mathbf{0} \tag{1}$$

where λ is the eigenvalue or load factor, $\mathbf{U}$ is the nodal displacement vector, $\mathbf{K}$ is the usual stiffness matrix of the structure, and $\mathbf{K}_f$ is the stiffness matrix of the same structure due to stress stiffening from an externally applied force f of arbitrary magnitude [2], [3]. That is, for a non-trivial solution to exist, the determinant of the multiplier matrix must be zero.

$$\|\mathbf{K} - \lambda \mathbf{K}_f\| = \mathbf{0}. \tag{2}$$

Once the eigenvalues are found, the critical buckling load f_c of the structure is given as follows.

$$f_c = \lambda_1 f, \tag{3}$$

where λ_1 is the lowest eigenvalue.

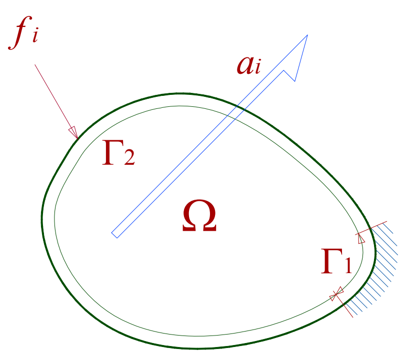

Figure 1 Schematic of the present problem – an object subject to both external force f_i and acceleration a_i.

For the current problem, as depicted in Figure 1, the deformable object is loaded with a reference force $f = \|f_i\|$ while being subject to a given acceleration motion $a_0 = \|a_i\|$. As a result, there would be two stress stiffening matrices due to the applied load and acceleration, $\mathbf{K}_f$ and $\mathbf{K}_{a0}$, respectively. It is our goal to determine the buckling load of the structure while it is under the given acceleration. Thus for the current problem an eigenvalue system to be solved may be given below.

$$[\mathbf{K} - \lambda(\mathbf{K}_f + \mathbf{K}_{a_0})]\mathbf{U} = \mathbf{0}, \tag{4}$$

After the eigenvalue problem is solved, the critical buckling load of the structure can be determined using Eqn. (3). Meanwhile, there would be a "critical acceleration" which in combination with the critical load would cause the structure to be in an unstable state. The acceleration under the critical condition a_c is no longer equal to the original acceleration a_0. Rather it is the one determined as follows.

$$a_c = \lambda_1 a_0. \tag{5}$$

Unless $\lambda_1 = 1$, $a_c \neq a_0$. Clearly, the above methodology does not provide the correct solution to the problem.

Consider the following constrained eigenvalue problem.

$$[\mathbf{K} - \lambda(\mathbf{K}_f + \alpha\mathbf{K}_a)]\mathbf{U} = \mathbf{0}, \tag{6}$$

where $\mathbf{K}$, $\mathbf{K}_f$ are the same matrices as before, $\mathbf{K}_a$ is the stress stiffening matrix using a reference acceleration a, and α is an *unknown* participation factor. One of the eigenvalues λ_k, typically λ_1, of the above eigenvalue problem is subject to the following condition / constraint.

$$\alpha a \lambda_k = a_0, \tag{7}$$

where a_0 is the given constant acceleration. Of concern is the buckling load f_c of the structure while the acceleration remains at a_0. Since a is a reference number, we may choose $a = 1$ for convenience.

Other constrained eigenvalue problem is given below [14], [12].

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^\mathbf{T} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{V} \end{Bmatrix} = \lambda \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{U} \\ \mathbf{V} \end{Bmatrix}. \tag{8}$$

Or,

$$\mathbf{AU} + \mathbf{B}^\mathbf{T}\mathbf{V} = \lambda\mathbf{CU}, \tag{9}$$

subject to

$$\mathbf{BU} = 0. \tag{10}$$

The above problem sees its applications in determination of the vibration modes of a structure when there are a few equality constraints imposed on the some nodal displacements in the model.

3. NUMERICAL ALGORITHM

For a given structure, the total stiffness matrix $\mathbf{K}$ can be readily formed first. The stress stiffening matrix $\mathbf{K}_f$ can be obtained by selecting an arbitrary f which remains the same throughout the following numerical scheme until the unknown α is determined. To obtain the stress stiffening matrix $\mathbf{K}_a$ due to acceleration, we may choose a = 1 for convenience. In the following numerical scheme, the value of the participation factor α is varying by an increment of choice. In essence, the acceleration is also varying by the same increment.

A trial-and-error method is presented to solve the constrained eigenvalue problem Eqns. (6) and (7). In the trapping scheme seen in Figure 2, the eigenproblem is solved using a series of guessed values of α_i. For each α_i, once the smallest eigenvalue λ_1 is found, the value a_i is calculated using Eqn. (7) $a_i = \lambda_1\alpha_i$. The computing cycle continues until the target value a_0 is trapped within the interval: $a_i \leq a_0 \leq a_{i+1}$.

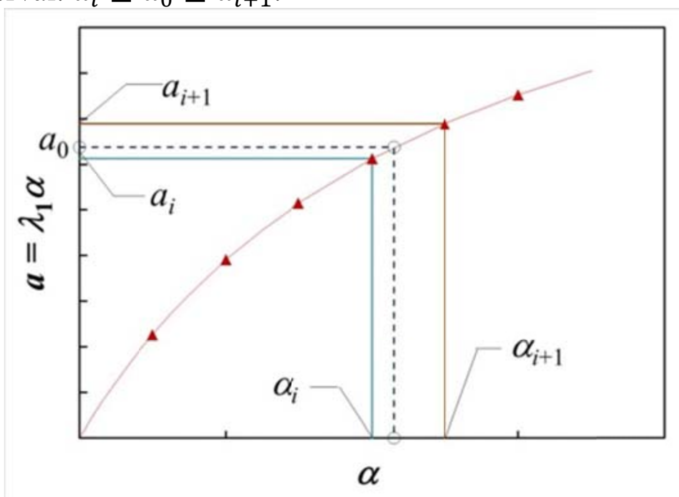

Figure 2 The trapping scheme for finding unknown α.

It is more convenient to introduce a natural coordinate ξ, $-1 \leq \xi \leq +1$. From the following linear interpolation, we can determine the natural coordinate ξ corresponding to the target value a_0.

$$a_0 = \frac{1}{2}(1-\xi)a_i + \frac{1}{2}(1+\xi)a_{i+1}. \tag{11}$$

Or,

$$\xi = (2a_0 - a_i - a_{i+1})/(-a_i + a_{i+1}). \tag{12}$$

Upon substituting this natural coordinate into the following interpolation equation, the unknown participation factor can be determined.

$$\alpha = \frac{1}{2}(1-\xi)\alpha_i + \frac{1}{2}(1+\xi)\alpha_{i+1}. \tag{13}$$

The eigenproblem Eqn. (6)is solved one last time using the participation factor found from Eqn. (13). The eigenvalue found together with the participation factor in Eqn. (13) constitute the solution to the constrained eigenvalue problem. The algorithm of this numerical scheme is given in the flowchart in Figure 3. In the following section, we use three examples to demonstrate the accuracy and efficiency of the algorithm presented here.

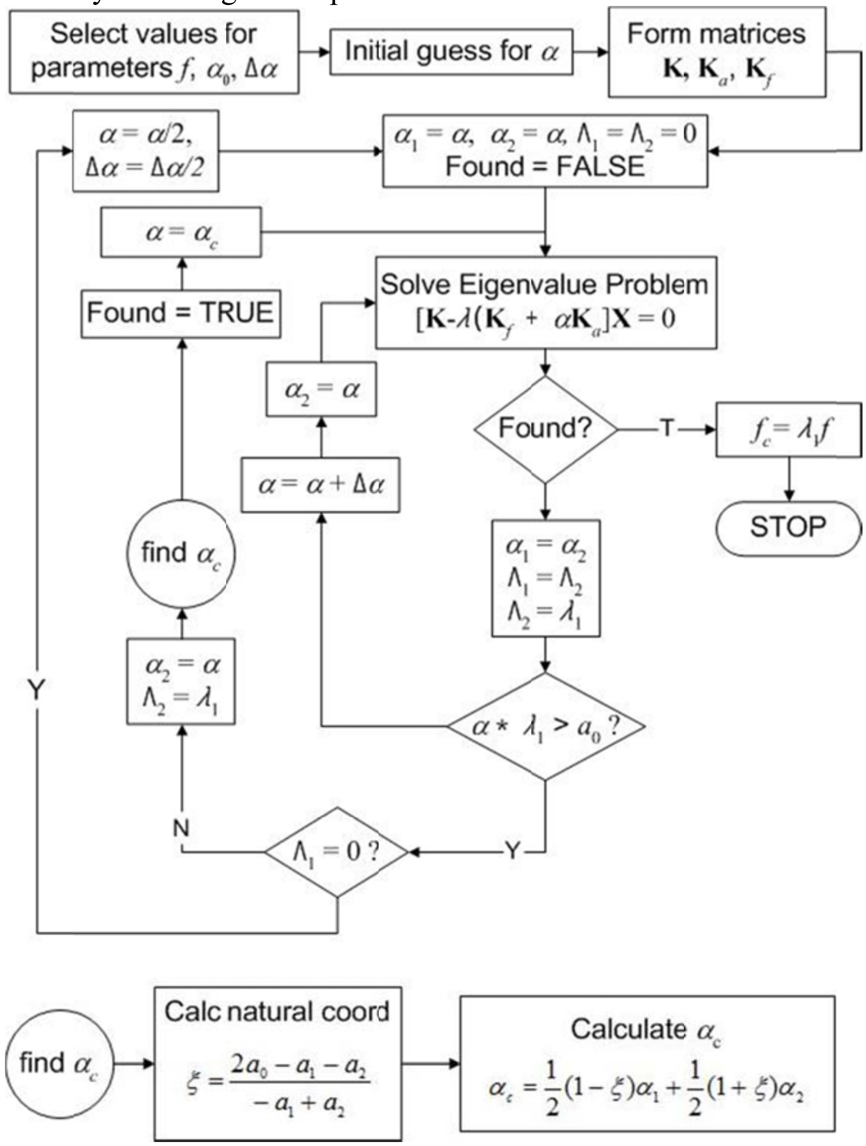

Figure 3 Flow chart for determination of buckling load.

4. APPLICATION EXAMPLES

All the examples presented in this section are two-dimensional; the algorithm can be extended to three-dimensional cases easily. A theoretical solution in approximate form exists for the first example, which serves as the guide for validating the accuracy of the proposed algorithm. In the other two examples, the purpose is to demonstrate the efficiency of the numerical algorithm. It is not intended to identify the worst case scenario.

4.1 Buckling of a Beam under Gravitational Force

As depicted in Figure 4, an elastic beam is subjected to a point force F at the upper free end as well as the gravitational pull. The theoretical solution of the buckling load is given approximately as [1].

$$F_{cr} \approx \frac{\pi^2 EI}{4L^2} - 0.3\rho g A L, \tag{14}$$

where EI is the flexural rigidity, ρ the density, A the cross-sectional area, and L the length of the elastic beam. In the numerical example, a wide-flange beam is used. The magnitude for the gravity is $g = 9.81$ m/s^2. Note that the beam would buckle due to its own weight if the following equation holds [1].

$$(\rho g A L)_{cr} \approx \frac{7.837 EI}{L^2}, \tag{15}$$

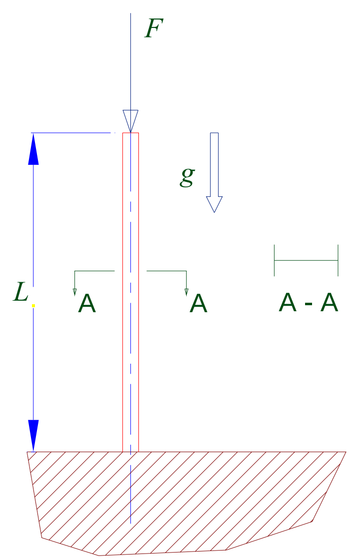

Figure 4 An elastic steel (E = 200 GPa, ρ = 7,870 kg/m^3) beam subject to gravitational force and an axial force F. Length of beam L = 5 m. For the wide-flange beam: cross-sectional area A = 158×10^{-6} m^2 and moment of inertia I_{zz} = 2.725×10^{-9} m^4.

In the finite element model, twenty five two-dimensional beam elements are used to model the vertical beam in Figure 4. Each node of the beam element has three degrees of freedom: two translational displacements and one rotation. Table 1 reveals a few calculation steps used to trap the desired acceleration g = 9.81 m/s^2 between the 4th and the 5th steps. Here, the reference load selected is F = 10 N. Therefore,

$$a_4 = 9.7455;\ a_5 = 10.3071.$$

From Eqn. (12)the natural coordinate corresponding to the gravitational acceleration g is:

$$\xi = -0.7701.$$

The participation factor determined through Eqn. (13) is:

$$\alpha = 2.7787.$$

Table 1 Numerical calculation of finding the buckling load for the elastic beam in Figure 4.

No.	F, N	α	λ	a, m/s^2
1	10	2	3.9121	7.8242
2		2.25	3.7813	8.5079
3		2.5	3.6588	9.1470
4		**2.75**	3.5438	**9.7455**
5		**3**	3.4357	**10.3071**

With the combination of F = 10 N and α = 2.7787, the eigenvalue problem Eqn. (6) gives λ_1 = 3.5311. Consequently, the beam is subject to the acceleration $a = \alpha\lambda_1$ = 2.7787×3.5311 = 9.812m/s^2, which is the gravitation acceleration now. And, the buckling load for the beam is: $F_{cr} = F\lambda_1 = 35.311$ N. The approximate theoretical solution to the problem according to Eqn. (14) is

$$F_{cr} \approx 35.141 \text{ N}.$$

The difference between the two solutions is less than 0.48%.

4.2 Buckling of a Truss Structure

Figure 5 shows a plane truss of a simplified crane. It has three point masses at three different locations. The mass m_1 at point A represents the mass of a counterweight, while $m_2 = m_3$ are the masses of the control unit of the crane at points D and E. The truss is constrained from translational movement at points A and B. Note that all members of the truss and the two cables are made of steel (E = 200 GPa, ρ = 7,870 kg/m^3). It is our intent to determine the maximum load W_c that the truss can carry at point F prior to buckling.

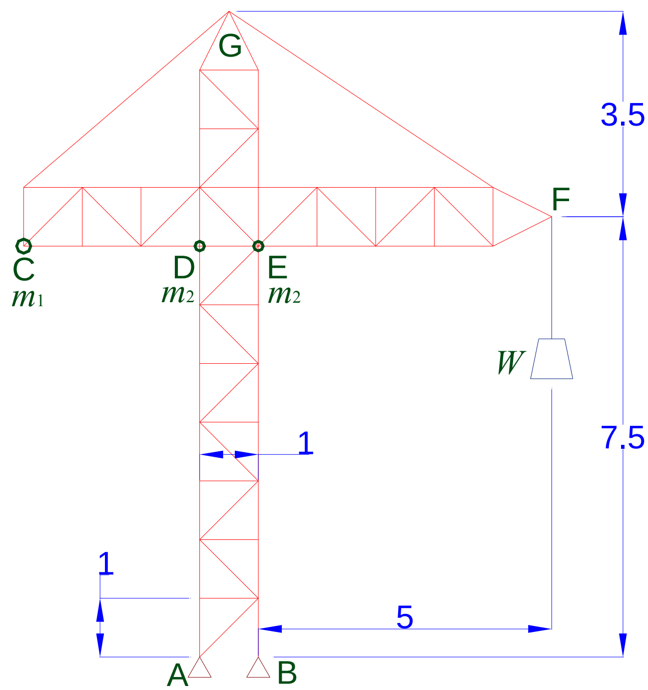

Figure 5 A plane truss with three concentrated masses m_1, m_2, $m_3 = m_2$ at points C, D, and E, carries a payload W. Given masses are m_1 = 1,000 kg, m_2 = 2,000 kg. Truss is fabricated by pinning bars with circular cross-section r_1 = 25 mm, and radius of cable r_2 = 25 mm.

In this example, 74 link elements are used for the bars and cables. Each node has two translational degrees of freedom. Three point elements are used to model the masses at points C, D, and E. In the numerical calculation shown in Table 2, the first value chosen for α overshoots the target acceleration $g = 9.81\text{m/s}^2$. Its value is decreased to $\alpha = 0.0011$ and a proper acceleration is realized. Therefore,

$$a_4 = 9.3238; \; a_5 = 10.0955.$$

Table 2 Numerical calculation of finding the buckling load for the plane truss in Figure 5.

No.	W, KN	α	λ	a, m/s^2
1	1.0	0.002	7738.3	15.4766
2		0.0015	7757.9	11.6369
3		0.0011	7773.7	8.5511
4		**0.0012**	7769.8	**9.3238**
5		**0.0013**	7765.8	**10.0955**

From Eqn. (12) the natural coordinate corresponding to the gravitational acceleration g is:

$$\xi = 0.260.$$

The participation factor determined through Eqn. (13) is:

$$\alpha = 0.001263.$$

The eigenvalue problem Eqn. (6) is solved one final time using W = 1.0 KN and α = 0.001263 which results in λ_1 = 7767.3. The downward acceleration the plane truss subject to is $a = \alpha\lambda_1$ = 0.001263 × 7767.3 = 9.810 m/s^2. And, the buckling load for the beam is:
$W_c = W\lambda_1 = 7767.3$ KN.

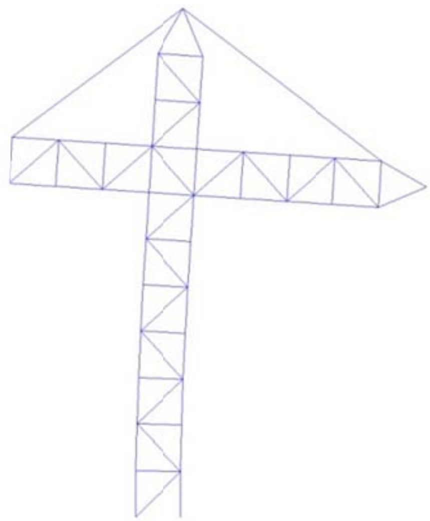

Figure 6 The 2D truss buckles in the first mode when W_c = 7767.3 KN.

The truss in the first buckling mode is shown in Figure 6. Note that upon solving the eigenvalue problem from Eqn. (1) using the stress stiffening matrix $\mathbf{K}_f$ of the truss structure stemming from the reference force F = 1.0 KN and the gravitational acceleration g = 9.810 m/s^2, the eigenvalue is λ_1 = 145.03. This indicates that the buckling load would have been $W_c = W\lambda_1 = 145.03$ KN, which is only 2% of the buckling load using the current algorithm. To cause the truss structure to buckle at this load the gravitational acceleration would have to be $a = g\lambda_1$ = 9.81 × 145.03 = 1422.72 m/s^2.

4.3 Buckling of a Plane Frame

In the third example, a traveling hoist installed on a plane frame is carrying a payload W. In addition to the mass of the hoist at point D, m_2, there is another mechanism at point C with mass m_1. The upper horizontal beam of the frame is of a C-shaped cross-section, while the vertical beams are of wide-flange. It is of interest to know the buckling load W_c of the frame when the hoist is located at point D.

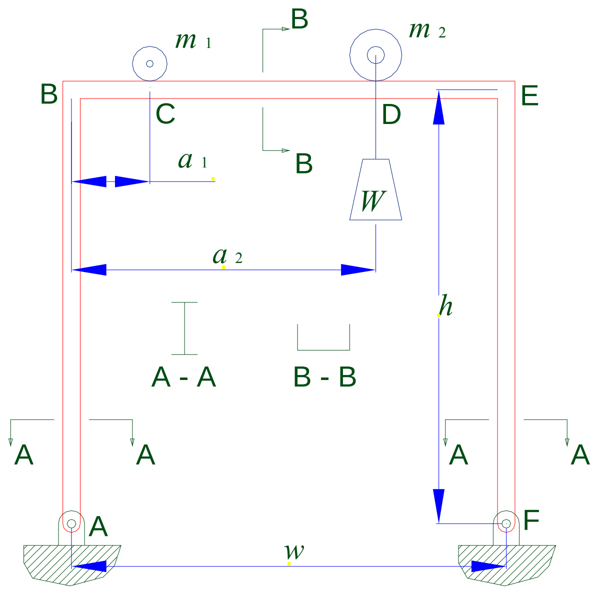

Figure 7 A plane frame with a stationary mass m_1 and a traveling hoist of mass m_2 carries a payload W. Given dimensions $h = 6$ m, $w = 4$ m, $a_1 = 1$ m, $a_2 = 3$ m, and masses are $m_1 = 100$ kg, $m_2 = 200$ kg. Cross-sections of beams, A-A: $A_1 = 484\times10^{-6}$ m^2 and $I_1 = 4.204\times10^{-8}$ m^4, and B-B: $A_2 = 384\times10^{-6}$ m^2, and $I_2 = 9.620\times10^{-8}$ m^4.

As indicated in Table 3 steps 4 and 5 trap the target acceleration g = 9.81 m/s^2 when the reference load selected is W = 1,000 N. Therefore,

$$a_4 = 9.6274;\ a_5 = 10.0128$$

From Eqn. (12) the natural coordinate corresponding to the gravitational acceleration g is:

$$\xi = -0.05241.$$

The participation factor determined through Eqn. (13) is:

$$\alpha = 10.9476.$$

Table 3 Numerical calculation of finding the buckling load for the plane frame in Figure 7.

No.	F, N	α	λ	a, m/s^2
1	1000	4	1.7874	7.1496
2		6	1.3904	8.3424
3		8	1.1377	9.1016
4		**10**	0.9627	**9.6274**
5		**12**	0.8344	**10.0128**

With the combination of W = 1,000 N and α = 10.9476, the eigenvalue problem Eqn. (6) gives λ_1 = 0.8974. It yields the beam being subject to the acceleration $a = \alpha\lambda_1$ = 10.9476×0.8974 = 9.824, which is the gravitation acceleration now. And, the buckling load for the frame is: $W_c = W\lambda_1 = 897.4$ N. It is seen from Table 3 the increment used for the calculation is $\Delta\alpha$ = 2.0. If $\Delta\alpha$ = 1.0 is used, the buckling load for the frame would be $W_c = 901.4$ N, which represents a 0.4% change. The deformed shape of the frame in the first buckling mode when $W_c = 901.4$ N is shown in Figure 8.

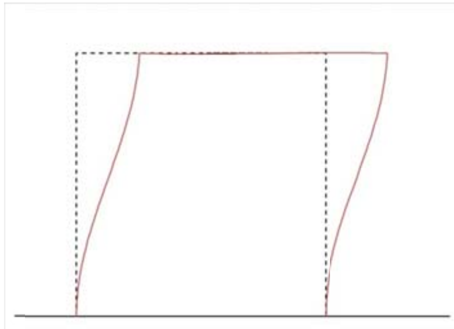

Figure 8 The plane frame in Figure 7 in the first buckle mode.

It is worth noting that, had the current algorithm been ignored, the buckling load of the frame would have been $W_c = 977.0$ N as a result of using α = 9.81 in Eqn. (6). It represents 8.4% higher than the critical load predicted using the current algorithm.

5. CONCLUSION

The determination of the buckling load of an elastic structure in the presence of gravitational force is formulated as an eigenvalue problem subject to an equality constraint correlating an unknown participation factor and the gravitational acceleration. A methodology of solving the constrained eigenvalue problem is presented. In the numerical algorithm, the eigenvalue problem is solved incrementally until the desired participation factor falls within an interval. Interpolation is employed to extract the accurate solution for the unknown. Three examples are used to demonstrate the accuracy of the numerical algorithm. Among them, one has an approximate theoretical solution. The solution predicted by the proposed algorithm is in excellent agreement with the theoretical solution. From the other two examples involving two-dimensional truss and frame, it is shown that the critical buckling loads predicted from the proposed algorithm are lower than those from the usual procedure by a relatively significant amount. The procedure involves some manual intervention and is laborious. It is necessary to develop an automatic numerical scheme for the problem in the future.

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