

LARGE DEFLECTION OF COMPOSITE LAMINATE THIN PLATES BY THE BOUNDARY ELEMENT METHOD

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Abstract. *Boundary-integral equations for large deflections of thin plates are presented. Quadratic boundary elements are used to discretise the boundary. Domain integrals that arise from non-linear terms are transformed into boundary integrals using the radial integration method. As a result, the obtained formulation does not demand domain discretization. For the solution of the non-linear system, the total incremental method is used. A numerical example is presented and comparisons with other numerical results are made to demonstrate the accuracy of the proposed method. The formulation illustrates the adaptability of the boundary element methods to non-linear problems.*

Keywords: *Boundary Element Method, Radial Integration Method, Thin Plates.*

1. INTRODUCTION

The boundary element method has already been established as an extremely efficient and effective technique in the analysis of linear and non-linear structural problems, as can be seen in the book of Aliabadi [1]. For linear problems, it is always possible to find a fundamental solution considering all terms of governing equations, producing formulations with pure boundary discretization. However, when non-linear differential equations are considered, fundamental solutions are found only to the linear portion of the equations. As a result, non-linear formulations have not only boundary integrals but also domain integrals. Thus, the advantage in reducing the dimensionality, as is the case in linear boundary element method, is lost in the non-linear case. Some procedures have been developed to avoid domain discretization, as can be seen in the works of Purbolaksono and Aliabadi [2] or Wen *et al.* [3,4]. Apart from numerous articles where the boundary element method is applied to non-linear isotropic problems, to the best of authors knowledge, it has not been found in literature a paper where the boundary element method is applied to a non-linear analysis of anisotropic materials.

The linear theory of Kirchhoff for thin plates is valid only when the transverse deflections are approximately less than the thickness of the plate. The extension to large deflections, where the non-linear terms are retained in the kinematic relationships, leads to a pair of coupled non-linear fourth order equations for the transverse displacement and the stress function for the in-plane stress resultants. Purbolaksono and Aliabadi [5] discuss and compare some approaches to compute large deflections of isotropic plates using the boundary

element method. Non-linear formulations demand iterative solutions that, depending on the chosen approach, is necessary to rebuild influence matrices for each load step. This kind of procedure gives better results. However, they are very time consuming. This work presents a boundary element formulation for the analysis of large deflections in thin plates.

2. BOUNDARY ELEMENT FORMULATION

The hypothesis of large deflections used in this work is:

- Deflections u_3 (displacements in the direction of axis x_3 , transversal to the plate) are large (many times bigger than the thickness h of the plate) but strains are still small when compared to the unity (changes in geometry can be disregarded).

Membrane stress resultants N_{ij} ($i, j = 1, 2$), i.e., stress due to extension, compression or shear on the medium plane of the laminate, integrated along the thickness, for a composite laminated plate, considering large deflections, are given by (Chia [7]):

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \left(\begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} u_{3,1}^2 \\ u_{3,2}^2 \\ 2u_{3,1}u_{3,2} \end{Bmatrix} \right) = \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}^{(l)} + \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}^{(nl)}, \quad (1)$$

where:

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}^{(l)} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} \end{Bmatrix} \quad (2)$$

and

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix}^{(nl)} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} u_{3,1}^2 \\ u_{3,2}^2 \\ 2u_{3,1}u_{3,2} \end{Bmatrix}, \quad (3)$$

indices (l) and (nl) stand for linear and non-linear components, respectively, A_{ij} ($i, j = 1, 2, 6$) are terms of the laminate extension stiffness matrix; u_i ($i = 1, 2$) are displacement in directions of axes x_1 and x_2 , u_3 is the deflection, i.e., displacement in the direction x_3 , normal to the plate surface.

In the absence of body forces, governing equations of an anisotropic plate under large deflections are written as (Shi [8]):

$$N_{ij,j} = 0, \quad (4)$$

$$D_{11}u_{3,1111} + 4D_{16}u_{3,1112} + 2(D_{12} + D_{66})u_{3,1122} + 4D_{26}u_{3,1222} + D_{22}u_{3,2222} = N_{ij}u_{3,ij}, \quad (5)$$

where $i, j, k = 1, 2$; D_{11} , D_{22} , D_{66} , D_{12} , D_{16} , and D_{26} are terms of the laminate bending stiffness matrix.

Integral equations for in-plane displacements, obtained by the reciprocity and Green theorems and equations (4) and (5), are given by (O'Donoghue e Atluri [6]):

$$\begin{aligned} c_{ij}u_j(Q) + \int_{\Gamma} t_{ij}^*(Q, P)u_j(P)d\Gamma(P) &= \int_{\Gamma} u_{ij}^*(Q, P)t_j^{(l)}(P)d\Gamma(P) \\ &+ \int_{\Omega} N_{ij}^{(nl)}(P)u_{ki,j}^*(P, Q)d\Omega, \end{aligned} \quad (6)$$

where $t_i^{(l)} = N_{ij}^{(l)}n_j$ stand for tractions on the boundary of the plate, parallel to the plane defined by $x_1 - x_2$; n_j is the unity vector normal to the boundary at the source point Q ; P is the field point; asterisks stands for fundamental solutions. Fundamental solutions of anisotropic plane elasticity can be found in Sollero and Aliabadi [9]. The constant c_{ij} is introduced in order to take into account that the source point Q can be in the domain, on the boundary, or outside the domain.

Resultant stresses for a point $Q \in \Omega$ are given by:

$$\begin{aligned} c_{ik}N_{kj}^{(l)}(Q) + \int_{\Gamma} S_{kij}^*(Q, P)u_k(P)d\Gamma(P) &= \int_{\Gamma} D_{kij}^*(Q, P)t_k^{(l)}(P)d\Gamma(P) \\ &+ \int_{\Omega} D_{kij}^*(P, Q)N_{k,ll}^{(nl)}(P)d\Omega \end{aligned} \quad (7)$$

where D_{ikj} and S_{ikj} are linear combinations of plane elasticity fundamental solutions.

The integral equation for anisotropic thin plate, obtained by the reciprocity and Green theorems, and by equation (5), is given by:

$$\begin{aligned} Ku_3(Q) + \int_{\Gamma} \left[V_n^*(Q, P)w(P) - M_n^*(Q, P)\frac{\partial w(P)}{\partial n} \right] d\Gamma(P) &+ \sum_{i=1}^{N_c} R_{ci}^*(Q, P)u_{3,ci}(P) \\ &= \sum_{i=1}^{N_c} R_{ci}(P)u_{3,ci}^*(Q, P) + \int_{\Gamma} \left[V_n(P)u_3^*(Q, P) - M_n(P)\frac{\partial u_3^*}{\partial n}(Q, P) \right] d\Gamma(P) \\ &+ \int_{\Omega} N_{ij}(P)u_{3,ij}^*(Q, P) d\Omega + \int_{\Gamma} [t_i(P)u_3^*(Q, P)u_{3,i}(P) \\ &- t_i(P)u_3(P)u_{3,i}^*(Q, P)] d\Gamma, \end{aligned} \quad (8)$$

where $\frac{\partial()}{\partial n}$ stands for derivative in the direction of the vector \mathbf{n} that is normal to the boundary Γ in the outward direction of domain Ω ; M_n and V_n are bending moment and equivalent shear force, respectively; R_c is the thin plate corner reaction; $u_{3,ci}^*$ is the deflection of the corner i ; constant K is introduced in order to take into account that the source point Q can be in the domain, on the boundary, or outside the domain. Fundamental solutions of anisotropic thin plate can be found in Albuquerque *et al.* [10].

An additional equation is necessary in order to have an equal number of unknown variables and equations. This equation is given by:

$$\begin{aligned}
& K \frac{\partial u_3}{\partial m}(Q) + \int_{\Gamma} \left[\frac{\partial V_n^*}{\partial m}(Q, P) w(P) - \frac{\partial M_n^*}{\partial m}(Q, P) \frac{\partial w(P)}{\partial n} \right] d\Gamma(P) \\
& + \sum_{i=1}^{N_c} \frac{\partial R_{c_i}^*}{\partial m}(Q, P) u_{3_{c_i}}(P) = \sum_{i=1}^{N_c} R_{c_i}(P) \frac{\partial u_{3_{c_i}}^*}{\partial m}(Q, P) \\
& + \int_{\Gamma} \left[V_n(P) \frac{\partial u_3^*}{\partial m}(Q, P) - m_n(P) \frac{\partial^2 u_3^*}{\partial n \partial m}(Q, P) \right] d\Gamma(P) \\
& + \int_{\Omega} u_3(Q, P) N_{ij}(Q, P) \frac{\partial u_{3,ij}^*}{\partial m}(Q, P) d\Omega + \int_{\Gamma} \left[t_i(P) u_3^*(Q, P) \frac{\partial u_{3,i}(P)}{\partial m} \right. \\
& \left. - t_i(P) u_3(P) \frac{\partial u_{3,i}^*(Q, P)}{\partial m} \right] d\Gamma, \tag{9}
\end{aligned}$$

where $\frac{\partial(\cdot)}{\partial m}$ is the derivative in the direction of the vector \mathbf{m} that is normal to the boundary Γ on the source point Q in the outward direction of the domain Ω , $t_i = N_{ij}n_j$ and $t_i^{(nl)} = N_{ij}^{(nl)}n_j$.

As can be seen, domain integrals arise in all integral equations. These domain integrals are given in the form:

$$P(Q) = \int_{\Omega} b(P) v^*(Q, P) d\Omega, \tag{10}$$

where b and v^* are generic body force and fundamental solution, respectively.

The body force is approximated over the domain Ω as a sum of M products between approximation functions f_m and unknown coefficients γ_m , that is:

$$b(P) = \sum_{m=1}^M \gamma_m f_m + ax + by + c \tag{11}$$

with

$$\sum_{m=1}^M \gamma_m x_m = \sum_{m=1}^M \gamma_m y_m = \sum_{m=1}^M \gamma_m = 0 \tag{12}$$

Domain integrals in the form of equation (10) approximated by the series given by (11) and (12) are transformed into boundary integrals by the radial integration method. Details of this method was given by Albuquerque and Aliabadi in [11] and will not be repeated here.

3. NUMERICAL RESULTS

Consider a clamped square plate, with side length a and thickness h , subjected to a uniform transverse load q . This plate is discretized using 4, 8 and 16 discontinuous boundary elements of equal length and 1, 9 and 25 uniformly distributed internal points. 15, 30 and 60 load steps are used in the total incremental method. The situation analysed is quasi-isotropic ($E_1/E_2 \cong 1$). Other material constants are given by $\nu_{12} = 0.316$ and $G_{12} = E_2/(1 + \nu_{12})$.

Results for the combination of all the discretization parameters are shown in the figures 3, 2 and 1 together with results obtained by Purbolaksono and Aliabadi [5] for the isotropic situation. The following normalizations are applied:

$$Q = \frac{qa^4}{E_2 h^4} \quad (13)$$

and

$$Z = \frac{u_{3c}}{h}, \quad (14)$$

where u_{3c} is the transversal displacement at the centre of the plate.

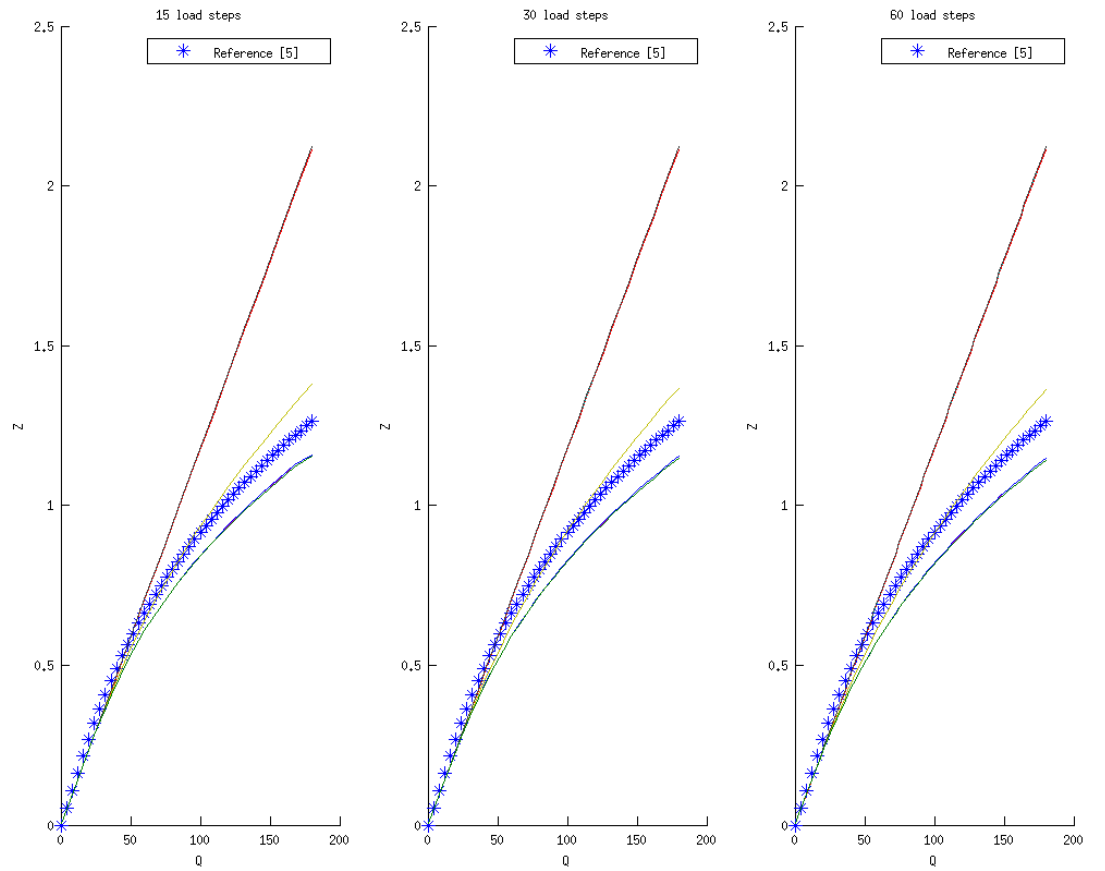


Figure 1. Normalized central deflection versus normalized load for the clamped square plate under uniformly distributed load.

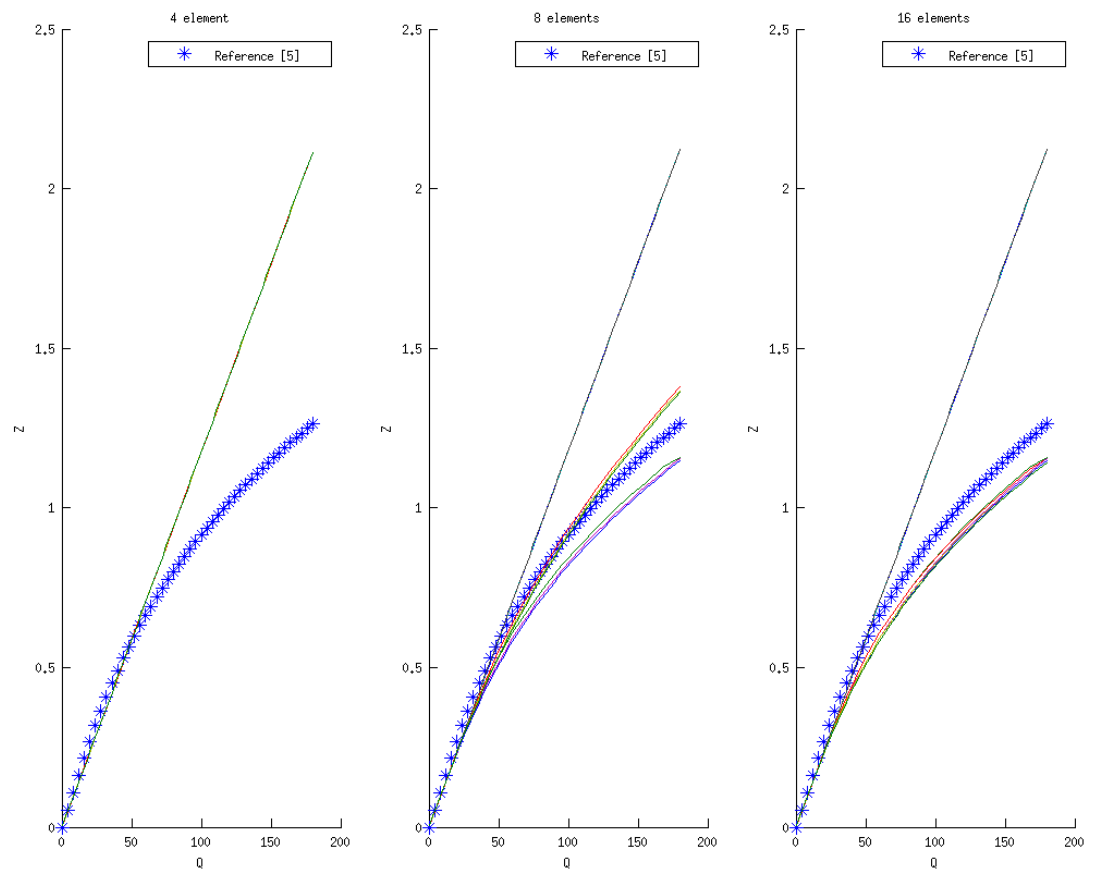


Figure 2. Normalized central deflection versus normalized load for the clamped square plate under uniformly distributed load.

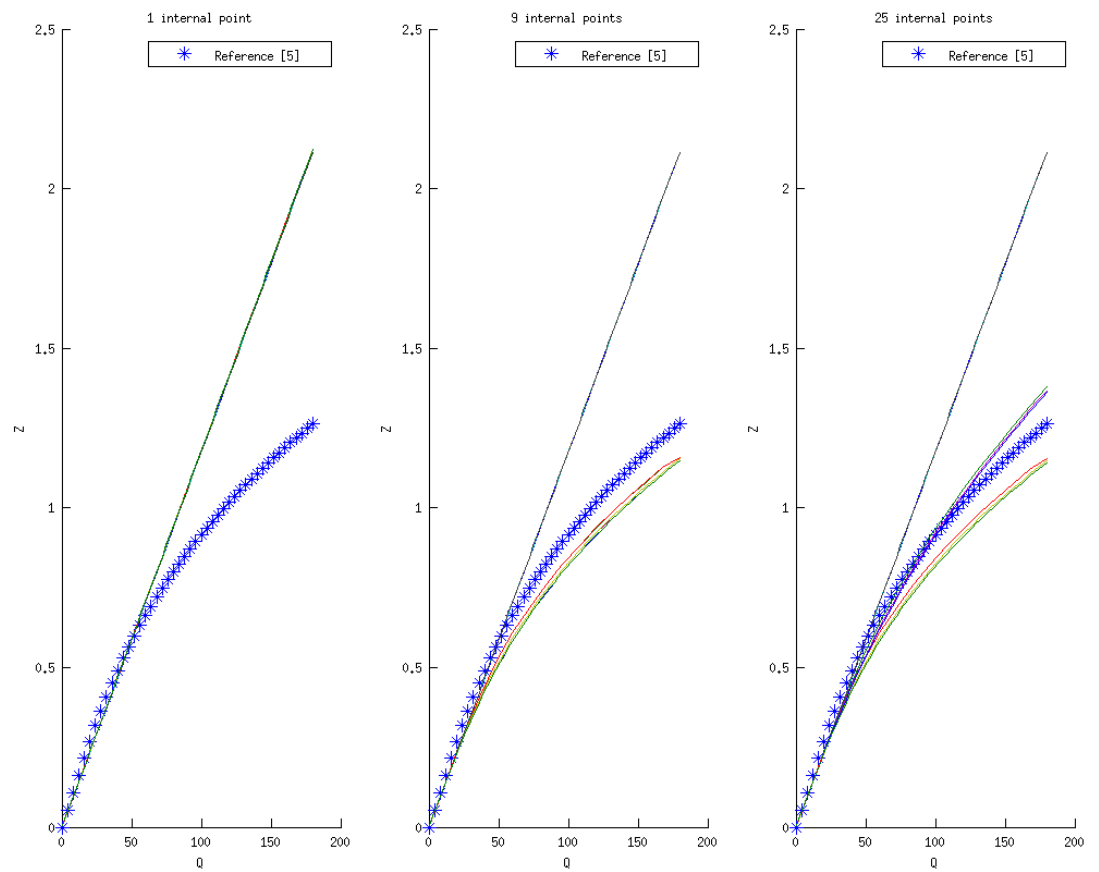


Figure 3. Normalized central deflection versus normalized load for the clamped square plate under uniformly distributed load.

We can easily see that for poor discretizations the non-linear effects are not taken into account and the response is linear. From Figures 3 and 1 we can conclude that only one internal and one element per side are not enough for the radial integration method to satisfactorily integrate the non-linear equations. However for superior values of discretizations, there is a good agreement between the isotropic results of Purbolaksono and Aliabadi [5] and the isotropic results obtained in this work.

4. CONCLUSIONS

This paper presented a boundary element formulation for the analysis of large deflections in thin plates. The formulation has a pair of coupled non-linear fourth order equations for the transverse displacement and the stress function for the in-plane stress resultants. Domain integrals that arise from non-linear terms are transformed into boundary integrals using the radial integration method. The numerical results presented good agreement with literature, which demonstrates the adaptability of the boundary element methods to non-linear problems.

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