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ROBUST OPTIMIZATION USING REDUCED-ORDER MODELING FOR NON-LINEAR STATIC TRUSS SYSTEM

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Abstract. *In this work a design optimization tool to obtain robust optimum designs of trusses under nonlinear conditions is described and implemented. The robustness measures considered here are the expected value and standard deviation of the function involved in the optimization problem. To calculate such quantities, we employ two nonintrusive uncertainty propagation analysis techniques that exploit deterministic computer models: Monte Carlo (MC) method and Probabilistic Collocation Method (PCM). When using these robustness measures combined, the search of optimal design appears as a robust multi-objective optimization (RMO) problem. To overcome the time consuming problem inherent in a RMO problem a model reduction technique using the proper orthogonal decomposition (POD) method will be employed to provide fast outputs for nonlinear analysis of trusses. A structural sizing optimization (SSO) algorithm incorporating such procedure in the structural and sensitivity stochastic analyses will be used to obtain efficient optimal trusses design. Optimization studies will be conducted for trusses problems considering different loads level, exploring the material plasticity. Comparisons will be conducted with the SSO approach via traditional FEM and via POD.*

Keywords: *Robust Optimization, Multi-Objective Optimization, Proper Orthogonal Decomposition, Nonlinear Static Problem, Probabilistic Collocation Method.*

1. INTRODUCTION

On the design of most engineering applications, the traditional optimization approach is to consider deterministic models and parameters. However, some degree of uncertainty in characterizing any real engineering system is inevitable. Unfortunately, the deterministic approach generally leads to a final design whose performance may degrade significantly or constraints can be violated because of perturbations arising from uncertainties. In this scenario a better target that provides an optimal design is one that gives a high degree of robustness. That is a design which is relatively invariant with respect to changes in uncertain parameters. The process to find such optimal is referred to as robust [design] optimization (RO) [1, 2].

Commonly uncertainty in material properties, variation in geometry, uncertainty in loading and boundary condition, etc. are included in the design process by introducing simplified hypothesis. The efficient way to consider the uncertainties is by computing the statistics of the responses involved. Here, we discuss two nonintrusive uncertainty propagation analysis that exploit deterministic computer models: Monte Carlo (MC) method and Probabilistic Collocation Method (PCM) [3]. These approaches consider the computational system (code) as a black-box, which returns a function values given an input vector.

Several robustness measures have been proposed in the literature, in particular, the expected value and standard deviation of the function involved in the optimization problem are considered here. When using these robustness measures combined, the search of optimal robust design appears as a robust multi-objective optimization (RMO) problem, as this involves more than one objective.

The computation of the Pareto frontier [4, 5] solutions is the adequate procedure when a multi-objective problem has to be solved. Efficient Pareto distribution has been obtained for two objectives problems by means of algorithms such as NBI (Normal-Boundary Intersection) [6], and NNC (Normalized Normal-Constraint) [7]. These two strategies are implemented in this work together with other commonly considered approaches in literature such as weighted sum method and min-max method. For more than two objectives the modified NBI procedure (NBIm), developed for this propose, is also considered.

As the generation of Pareto points and the uncertainty analysis could be very costly, approximation techniques based on reduced-order modeling (ROM) approach are also incorporated in our procedure via proper orthogonal decomposition (POD) method [8, 9]. Here the POD method will be employed to provide to the optimizer, fast nonlinear response calculations for the trusses. Such technique approximates the numerical model by reducing the total number of degree of freedom of the original problem (high fidelity (HF) model). General reduced-order models are obtained by projecting the HF model in some low order basis. The POD is a ROM that, basically, projects the problem into a subspace formed by a optimum orthonormal basis functions, in the sense that it consider the most significant shape (greatest variance) of the output subspace. The process encompass two stages: The offline stage is done once to compute the basis of the projection. After completed, this basis is used in the online stage to obtain the approximated results.

A Structural sizing optimization (SSO) algorithm incorporating such procedure in the structural, sensitivity and probabilistic analyses will be used to obtain efficient optimal trusses design under nonlinear conditions. Optimization studies will be conducted for trusses problems considering different loads level, exploring the material plasticity. Comparisons will be conducted with the SSO by traditional FEM and by POD.

2. NONLINEAR STATIC ANALYSIS

The nonlinear static analysis employed in this work, will consider only the nonlinear stress-strain relation (material nonlinearity). In this sense, the displacements and internal forces relationship is nonlinear. The objective is to find the solution field (displacement) so that the internal forces equals the external forces

$$\mathbf{F}_\mathrm{i}(\mathbf{u}) = \mathbf{F}_\mathrm{e} \tag{1}$$

The iterative procedure generally used to obtain solution of plastic analysis is the Newton-Raphson (NR) method. The NR method iteratively approximates the nonlinear equation by a linearization in the current point (solution). In the plastic analysis it can be formulated for the kth iteration as

$$\mathbf{F}_\mathrm{i}\left(\mathbf{u}^k\right) + \frac{d\mathbf{F}_\mathrm{i}}{d\mathbf{u}}\left(\mathbf{u}^k\right)\Delta\mathbf{u}^k = \mathbf{F}_\mathrm{e} \;\text{ or }\; \mathbf{K}\mathrm{t}\Delta\mathbf{u}^\mathrm{k} = \mathbf{R} \tag{2}$$

in which $\mathbf{K}t = \frac{d\mathbf{F}_\mathrm{i}}{d\mathbf{u}}\left(\mathbf{u}^k\right)$ is the tangent stiffness matrix, and $\mathbf{R} = \mathbf{F}_\mathrm{e} - \mathbf{F}_\mathrm{i}\left(\mathbf{u}^k\right)$ is the residual load vector.

The iterative technique on its own can only provide a single 'point solution'. In practice, we will often prefer to trace the complete load/deflection response (equilibrium path). To this end, it is useful to combine the incremental and iterative solution procedures. The 'tangential incremental solution' can then be used as a 'predictor' which provides the starting solution, for the iterative procedure. A good starting point can significantly improve the convergence of iterative procedures and increases the possible incremental load step. Indeed it can lead to convergence where otherwise divergence would occur [10].

There are various incremental load step methods. A constant increment is considered here, in which the increment is proportional to the pressure at the yield stress. The increment (proportional factor) used in the analysis procedure will be specified in the Application section.

3. REDUCED ORDER METHODS

The large scale models require high computational costs. The reduced order method attempt to approximate high dimension system in a low dimension space. Most of ROMs project the high dimension system into a lower dimension space [11]. A simple example is the discrete system equation:

$$\mathbf{K}\mathbf{u} = \mathbf{F}\;,\;\; \mathbf{K} \in \mathbf{R}^{n,n} \text{ and } \;\mathbf{u}, \mathbf{F} \in \mathbf{R}^{n} \tag{3}$$

This system can be project into a subspace W of base Z $\in \mathbf{R}^{n,w}$ with $w < n$. As a result, the problem becomes

$$\mathbf{Z}^T\mathbf{K}\mathbf{Z}\mathbf{u} = \mathbf{Z}^T\mathbf{F}\text{ , or simply }\; \mathbf{K}^W\alpha = \mathbf{F}^W \tag{4}$$

where $\mathbf{K}^W = \mathbf{Z}^T\mathbf{K}\mathbf{Z} \in \mathbf{R}^{w,w}$, $\mathbf{F}^W = \mathbf{Z}^T\mathbf{F} \in \mathbf{R}^w$ and the vector $\alpha \in \mathbf{R}^w$, in which $\mathbf{Z}\alpha = \mathbf{u}^W \approx \mathbf{u}$ is the linear coefficient of the approximation output vector $\mathbf{u}^W$. Thus, the space W and consequently the base vectors Z are fundamentals to the efficiency of the method and they commonly represent the differences between the ROMs.

3.1. Proper orthogonal Decomposition

POD is a ROM that, basically, project the problem into a subspace formed by a optimum orthonormal basis functions, in the sense that it considers the most significant shape (greatest variance) of the output (**u**) subspace. A practical way to obtain these vectors is computing a set of **u** vectors for various system configurations (times, design variables, parameters, loads steps,) then, a singular value decomposition (SVD) is performed and to compute the eigenvalues of the covariance matrix of the outputs [12, 8]. This procedure is related to the method of snapshots. The method of snapshots was introduced by Lawrence Sirovich in 1987 [13] as a way to reduce the computational requirements of the POD basis. The snapshot matrix $\mathbf{X}$ can be written as:

$$\mathbf{X} = \left[\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^m\right] \tag{5}$$

the number of snapshots m is assumed to be sufficiently large for represent the field of solutions $\mathbf{u}$.

This POD basis can be obtained directly through a SVD of the snapshot matrix $\mathbf{X}$.

$$\mathbf{X} = \mathbf{S}\mathbf{D}\mathbf{V}^{\mathrm{T}} \text{ where } \mathbf{X}\mathbf{V} = \mathbf{S}\mathbf{D} \text{ and } \mathbf{X}^{\mathrm{T}}\mathbf{S} = \mathbf{V}\mathbf{D}^{\mathrm{T}} \tag{6}$$

in above equation $\mathbf{D}$ is related to the eigenvalues such as $diag(\mathbf{D}) = [\Lambda_1, \Lambda_2, \ldots \Lambda_n]$, $\Lambda_i = \sqrt{\lambda_i}$, and $\mathbf{V}$ are eigenvectors of the autocorrelation matrix $\mathbf{X}$.

To compute the POD basis, the first w eigenvalues such that

$$1 - \sum_{i=1}^{w} \bar{\lambda}_i < tol_\lambda, \text{ in which } \bar{\lambda}_i = \frac{\lambda_i}{\sum_k \lambda_k} \tag{7}$$

must be found. In the equation 7, the tol_λ parameter is the tolerance related to the energy error in the POD approximation. Note that $[w < m < n]$. After the number w of significant singular components be determined, the POD basis is computed

$$\mathbf{Z} = \mathbf{S}^w \tag{8}$$

where the upper w index, indicate the first w vectors (column) of the matrix.

To proceed the POD in the solution of the nonlinear structural analysis, the standard iterative equation (2) has to be changed, so that the displacement vector **u** is the unknown, rather than $\Delta\mathbf{u}$, following to

$$\mathbf{K}_t\left(\mathbf{u}_k - \mathbf{u}_{k-1}\right) = \mathbf{R}_k \text{ or } \mathbf{K}_t\mathbf{u}_k = \mathbf{R}_k + \mathbf{K}_t\mathbf{u}_{k-1} \tag{9}$$

This is due to the fact that the displacement vector **u** (correlated) is easier to approximate than the vector $\Delta\mathbf{u}$ (uncorrelated).

As can be seen, the POD has equivalence to the principal component analysis (PCA), the singular value decomposition (SVD) and Karhunen-Love (KL) decomposition, and has been widely and successfully applied in various disciplines, including fluid mechanics, static and dynamic structural mechanics, oceanography, statistics, economics, image processing, etc [14].

4. PROBLEM FORMULATION

The deterministic approach for optimization problem can leads to a final design whose performance may be very sensible to parameters variation. The Robust Optimization (RO) considers the uncertain to leads a design less susceptible to variability on uncertain parameters (U). In this work, two objective controls will be considerate: the mean and the standard deviation of a selected output function. This lead to a multi-objective optimization (MO) problem which is mathematically formulated as [1].

$$\min \mathbf{F}(U, \mathbf{x}) = \left[\ E\left(f(U, \mathbf{x})\right), \quad \sigma\left(f(U, \mathbf{x})\right) \ \right] \tag{10}$$

subject to:

$$\begin{array}{ll} g_i\left(U, \mathbf{x}\right) \leq 0 & i = 1, ...m \\ h_j\left(U, \mathbf{x}\right) = 0 & j = 1, ...\ell \\ x_k^\ell \leq x_k \leq x_k^u & k = 1, ...n_{dv} \end{array} \tag{11}$$

In which $E(*)$ is the expected value, $\sigma(*)$ is the standard deviation, F is the selected output and $\mathbf{x}$ is the design variable vector. The MO problem presented above is solved using the techniques described in Section 6.

5. STATISTICS CALCULATIONS

Assuming U as a random variable, any function $f(U)$ will be random, with its specific probability density function (PDF) $P(U)$. The expected value of $f(U)$, called mean of $f(U)$, can calculated as [15]:

$$E\left[f(U)\right] = \bar{f} = \int_{-\infty}^{\infty} f(U)P(U)dU \tag{12}$$

and its variance $\sigma_f^2 = \sigma[f(U)]^2$

$$\sigma_f^2 = E\left[\left(f(U) - \bar{f}\right)^2\right] = \int_{-\infty}^{\infty} \left(f(U) - \bar{f}\right)^2 P(U)dU \tag{13}$$

in which σ_f is the standard deviation.

In the present work two methodologies are employed for statistics calculations of several responses. They are Monte Carlo method and Probabilistic collocation method. Both methodologies are described in the following subsections.

5.1. Monte Carlo Method

The MC method is the most popular non-intrusive method and can be used for any problem related to uncertainty propagation [16]. Given the joint probability distribution function of the involved random variables, the MC method can be applied for approximated calculations of the statistics response of a particular quantity, including its distribution, with an arbitrary error, as long a sufficient number of samplings points is given. This approach has also been used as a ¨benchmark¨to validate other techniques for statistics calculations. In this method the functions $f(\mathbf{U})$ of interest are calculated in several random points U^k , generated taking into account their probability distribution $P(U)$, then the integrals of Eqs. 12 and 13 are respectively approximated as

$$\begin{aligned} \bar{f} \approx \bar{f}_{MC} = \tfrac{1}{m} \sum_{i=1}^{m} f(\mathbf{U}_{(i)}) \\ \sigma[f(\mathbf{U})]^2 = \sigma_f^2 \approx \hat{\sigma}_f^2 = \tfrac{1}{m-1} \left[\sum_{i=1}^{m} \left(f(\mathbf{U}_{(i)})^2 \right) - m\bar{f}_{MC}^2 \right] \end{aligned} \tag{14}$$

In which m is the number of sampling points, $\bar{f}_{MC}$ is the MC approximation for the mean values of $f(U)$ and $\hat{\sigma}_f$ is the MC approximation for the standard deviation.

5.2. Probabilistic Collocation Method

The basic idea of PCM is to approximate the function $f(U)$ by polynomial functions and to evaluate the integrals of Eqs. 12 and 13 by Gaussian quadrature. Gaussian quadrature is based on the concept of orthonormal polynomials. These concepts are briefly described here.

In the numerical integration by Gaussian quadrature for integrals of the form

$$F = \int f\left(U\right) P\left(U\right) dU \tag{15}$$

The function $f(U)$ is approximated by a polynomial of order $2n-1$ as follows [17]

$$f(U) \approx \hat{f}(U) = \left(\sum_{i=0}^{n-1} b_i h_i(U) \right) + h_n(U) \left(\sum_{i=0}^{n-1} c_i h_i(U) \right) \tag{16}$$

for $i = 1 \ldots n$ in which b_i and c_i are the coefficients of the approximation, to be obtained, and $h_i(U)$ are polynomials of order i from a orthonormal basis with respect to the weight function $P(U)$.

The statistics evaluations defined in Eqs. 12 and 13 via PCM is a direct application of Gaussian quadrature in which the PDF is the weighting function. Hence, by orthonormality, the approximated Gaussian quadrature integral Eq. (15) can be expressed as follows

$$F \approx b_0 h_0 \int_F P\left(U\right) dU = b_0 \tag{17}$$

To find the coefficients b_i and c_i of the Eq. (16) would be necessary to evaluate the function $f(U)$ in 2n points. However, as the integral presented in Eq. (17) does not depend on the coefficients c_i, it is required the calculations of function $f(U)$ only at the n roots (U^*) of $h_n(U)$, in this way canceling the second part of Eq. (16), as $h_n(U^*) = 0$. For more details concerning coefficients evaluations see [3].

The orthonormal polynomials are defined for each PDF and the roots (U^*) of each polynomial $h_i(U)$ are the quadrature points or integration point. Solving the approximation of Eq. (16) to find b_0, it follows that the mean value and, analogously, the standard deviation of an output of interest are approximated by PCM as

$$\bar{f}_{PC} = \sum_{i=1}^{n} P_i f(U^{k*})$$
$$\hat{\sigma}^2_{PC} = \sum_{i=1}^{n} P_i f(U^{k*})^2 - \bar{f}^2_{PC} \quad (18)$$

in which $P_i, : i = 1 \ldots n$, are the weight coefficients and U^* the integration points, calculated once PDF is given.

6. MULTIOBJECTIVE STRATEGIES

Pareto optimality concept [4] is used here to obtain MO solutions. The Pareto minima, are points x_p which for no other point x exist such that:

a) $f_k(x) \geq f_k(x_p)$ for $k = 1, \ldots, nobj$

b) $f_j(x) < f_j(x_p)$

for one objective function (f_j) at least. The discussions about this concept can be found in detail elsewhere [18, 6, 5].

Using the Pareto concept, the designer has to identify as many Pareto points as possible. These points can be used to construct a point-wise approximation to the Pareto front.

There are several techniques to obtain the set of Pareto minima. In this work we will consider the so-called objective weighting sum (WS) method, Min-Max method, the normal boundary intersection (NBI) method [6], and the normalized normal-constrain (NNC) method [7]. Currently, in literature, the later two strategies are pointed to have more success to obtain the Pareto curves. Such techniques are discussed in detail elsewhere [5].

7. APPLICATION

A plane truss will be optimization here, considering material nonlinearity under static load conditions. The geometric configuration and boundary conditions are presented in Figure 1, where the total number of degrees of freedom is 1210 [19].

The points of the stress-strain curve considered are illustrated in Table 1.

Table 1. Stress-Strain Curve.

Stress	Strain (KN/cm^2)
0.0025	51.750
0.0037	62.100
0.0050	72.450
0.0100	82.800
0.0175	93.150
0.0350	103.500
0.0750	113.850

Figure 1 shows the random variables (**U**) and the design variables (**x**). Thus, two random variables are considered: the vertical load on the top of the structure (U_1) and the horizontal load on the top-left side of the structure (U_2). The first one (U_1) has a log-normal

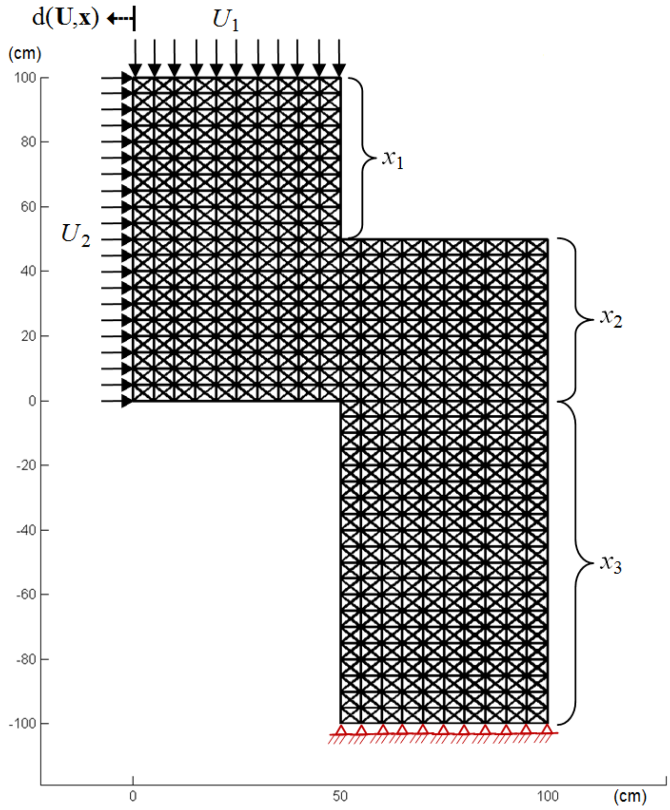

Figure 1. Structure and problem definition.

distribution with mean $\mu_1 = 4$KN/cm and standard deviation $\sigma_1 = 2$KN/cm, the second random variable (U_2) has a normal distribution with mean $\mu_2 = 0$ and standard deviation $\sigma_2 = 1$KN/cm. Three designs variables are considered, which are the cross section area of the bars of three regions, as shown Figure 1. The initial cross section areas (designs variables) are equal to one and the design variables are bounded by $0.1 \leq x \leq 10$.

The robust optimization problem can be formulated as:

$$\begin{array}{l} \min \left[\; \bar{d}(\mathbf{U},\mathbf{x}) \quad \sigma_d(\mathbf{U},\mathbf{x}) \; \right] \\ \mathrm{subjectto}: \\ vol^*(\mathbf{x}) \leq 1 \\ x_l \leq x_i \leq x_u \; , i = 1 \ldots n \end{array} \tag{19}$$

in which $d(\mathbf{U},\mathbf{x})$ cm is the horizontal displacement in the top left corner of the structure, see Figure 1. The vol$^*()$ is the relative volume of the structure, i.e., the current volume divided by the initial volume (vol(x_0) = 12371.1 cm^3), x_l and x_u are the lower and upper boundary of the design variables, respectively. The load increment used during the nonlinear analyses was Py/10, in which Py is the load level that lead to the yield stress.

A 1210x246 snapshot matrix (**X**) was obtained through the analysis via FEM of 30 different cases (considering different values for design variables and random variables). For a required tolerance of $tol_\lambda = 10^{-5}$, the size of POD basis generated was $w = 50$, for more details refer to [19]. To the statistical evaluation via PCM, in the initial design, the computational time consuming when using POD and fully FEM was 0.29s and 0.98s, respectively.

Also for the initial design, the PCM solution for different approximation degrees was confronted with the MC solution using 10^5 integration points. The PCM considering a 3x3 integration points grid (approximation of $5th$ degree) achieves a relative difference to the MC solution of 10^{-3}. The computational time consuming via MC was about 10^4 times greater than via PCM. Thus, the RMO procedure via PCM considering 9 collocation points was used.

The RMO problem was solved using the PCM approximations to evaluate the statistics of the structure. The nonlinear analysis were performed via POD reduced order model (for $w = 50$). The Pareto points obtained via the various MO methods cited here, are shown in Fig. 2. As expected, the results via NBI and NNC agree closely. The better Pareto points distribution was obtained by these two methods.

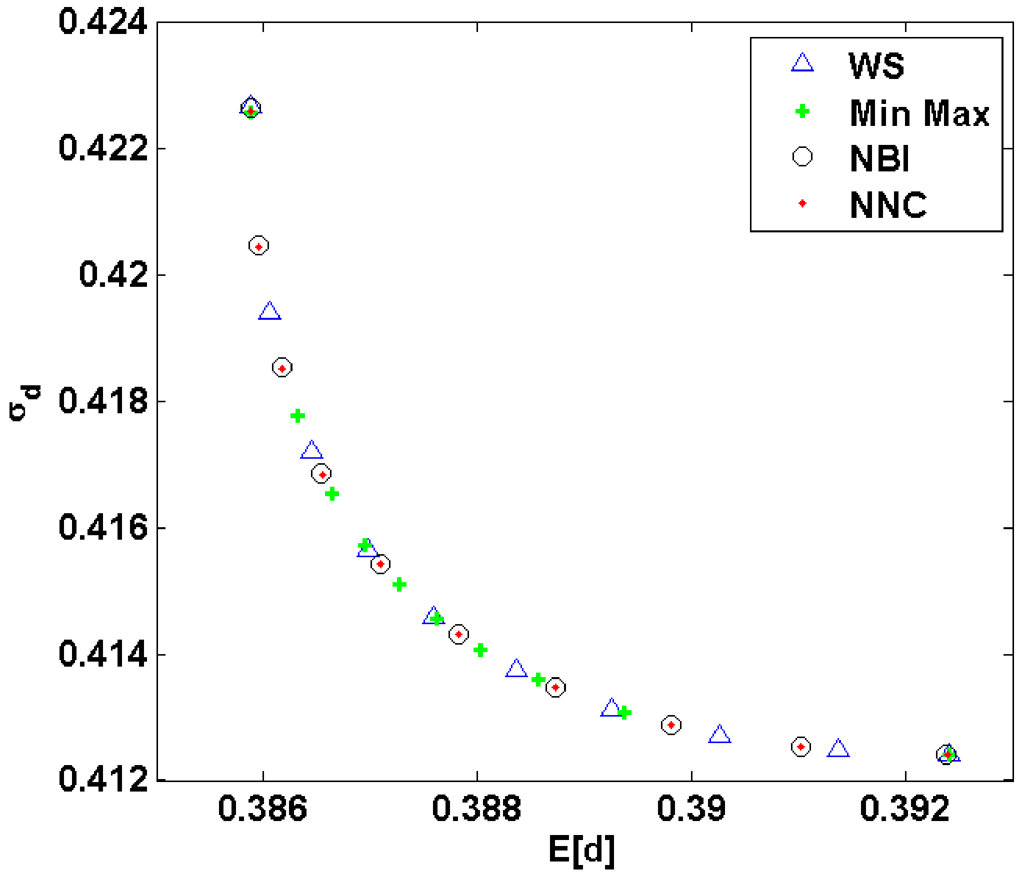

Figure 2. Pareto solutions via different MO methods.

The four multi-objective optimization performance, are shown in Table 2. In that table, the number of function evaluations (F. Count) is the total number of statistic analysis evaluated to obtain the Pareto points. the uniformity distribution of the Pareto points parameter (Evness), that appears in table 2, indicates the quality of the distribution of the points, the closer to zero the better [7, 5]. The most efficient method in this example was the NBI method, about 2 times faster than the others.

Table 2. Optimization performance considering PCM with POD methods.

MO Method	Time (s)	F Count	Evness
WS	50.7	210	0.300
MinMax	119.2	299	0.896
NBI	27.3	143	0.109
NNC	58.6	181	0.109

8. CONCLUSIONS

In this paper a RMO problem was solved using PCM to evaluate the statistics (1st and 2nd statistical moment) of the response of a truss under nonlinear condition and several multi-objective optimization techniques (Ws, Min-Max, NBI and NC methods) were used to obtain Pareto solutions. A POD algorithm was implemented to approximate nonlinear FEM analysis, considering the material nonlinearity. A good agreement in the nonlinear structural results and the optimum points by the fully FEM and POD analysis was achieved. For an error tolerance of 10^{-5} a basis of just 50 components is used to approximate an output of 1210 components. In summary:

- The computational time consuming to obtain results by POD was $1/3$ of the computational time consuming to obtain results via FEM, demonstrating the effectiveness of the method;
- The statistics computation via PCM require about 10^3 times less integration point than the MC method, for the same relative error;
- For the bi-objective example, the most efficient MO method was the NBI method, about 2 times faster than the others and obtaining evenly distributed Pareto points.

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