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HERMITE FINITE ELEMENTS FOR FLUID FLOW

D. N. Brandão[3,4], G. Buscaglia[1], M. Kischinhevsky[3], V. Ruas[1,2]

[1] Institute of Mathematical Sciences & Computing, USP-University of São Paulo/SC, Brazil

[2] Institute Jean Le Rond d'Alembert, University Pierre & Marie Curie, Paris 6, France

[3] Department of Computer Science, UFF-Federal University Fluminense, Niterói, RJ, Brazil

[4] Department of Computer Science, CEFET, Nova Iguaçu, RJ, Brazil

Abstract. *Traditionally Hermite finite element methods have been used to solve PDE's of order higher than two. The goal of this work is to show that this technique is very useful for solving second order PDE's too, whenever the direct determination of quantities expressed in terms of the derivatives of the solution, such as curvatures and fluxes is necessary. Emphasis will be given to applications of these discretization methods in the framework of flows on curved manifolds and flows in highly heterogeneous porous media. Corresponding error analyses and illustrative numerical results are given.*

Keywords: *Finite elements, Hermite, incompressible flow, porous media.*

1. Introduction

Hermite finite element methods have mostly been used to solve fourth order elliptic or parabolic equations, modelling a certain number of problems in Solid and Fluid Mechanics. Among well-known applications in this framework lie plate bending modeling and the incompressible Navier-Stokes equations in terms of the stream function or the vector potential. This is because this kind of methods using derivatives as degrees of freedom, seems quite natural to ensure an acceptable conforming representation of the solution. The fact that second order problems do not require the use of this kind of approach to attain the same goal, is probably the reason why the use of Hermite finite element methods in this context has been rather overlooked so far. Instead, whenever the direct representation of derivatives, fluxes or quantities alike is required, most authors consider the use of natural mixed formulations, in which the main unknown function and such related quantities are the multiple unknown fields of a system equivalent to the original equation. However it is well-known that, if this mixed problem is recast in the equivalent variational form of the standard Galerkin type, the stability issue of its discrete finite element analogue becomes critical. This limits the choice of performing methods to solve the system numerically. For this reason alternative non standard mixed formulations were proposed since the eighties, most known as stabilized mixed formulations. These usually involve added terms to the standard Galerkin formulation depending on stabilizing parameters, whose control is not always so obvious or easy to deal with in practical situations.

The main purpose of this work is to show that Hermite interpolation provides a valid alternative to represent not only the solution itself, but also related quantities expressed in terms of its partial derivatives, without any need to consider stabilizing formulations. This is because in general the resulting finite element method is to be used in connection with the natural Galerkin formulation as a conforming method. This means that the methods inherits the stability properties of the continuous problem, and hence its convergence can be easily established. Moreover its implementation can be achieved in a straightforward manner.
More specifically we consider here two applications of Hermite finite element methods. The first one is the simulation of flows in porous media taking as a model Darcy's law. Referring to [1] for the function space notations, in its simplest form this model corresponds to the following equation:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } -\nabla \cdot [\mathcal{K}\nabla u] = f \text{ in } \Omega \tag{1}$$

where Ω is a bounded domain of $\Re^N$, $N = 2, 3$, f is given in $L^2(\Omega)$ and $\mathcal{K}$ is a tensor assumed to be symmetric and positive definite but not necessarily constant.

We also show in this work that the Hermite interpolation is particularly well-suited to represent the velocity field of a viscous incompressible fluid in the framework of a finite element solution. Again for the sake of conciseness we study more particularly the case of the stationary Stokes system, namely, the problem of finding a velocity field $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and a pressure $p \in L_0^2(\Omega)$ such that,

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega. \end{cases} \tag{2}$$

where μ is the fluid's kinematic viscosity, $\partial\Omega$ is the boundary of the flow domain Ω, $\mathbf{f}$ is a given force field in $L^2(\Omega)$, and $\mathbf{g}$ is a prescribed velocity on the boundary satisfying the zero global flux condition on $\partial\Omega$. The notation $L_0^2(\Omega)$ is used to represent the subspace of $L^2(\Omega)$ of functions having zero integral in Ω.

2. Methods for flow in porous media

The first method introduced in [9] for three-dimensional problems and in [10] for two-dimensional ones, referred to here as HP_2, is based on a Hermite representation of the unknown function u with complete quadratics defined in each element of a triangular or tetrahedral mesh. The second method based on a Hermite representation of u by means of incomplete quadratics in each N-simplex, was introduced in [11]. It can be viewed as a modification of the lowest order Raviart-Thomas mixed element [8] known as RT_0, leading to better convergence properties, though at equivalent cost. We refer to it hereafter as HRT_0.
Convergence results were derived for the Hermite methods under study, in connection with relevant model problems.
The presentation of the Hermite finite element methods is followed by a series of comparative numerical studies, in which their performance is confronted with the one of several well-established techniques to solve the diffusion equation and also the time-dependent convection-

M $\longrightarrow$	8	16	32
HRT_0 $\longrightarrow$	0.53309×10^{-3}	0.13400×10^{-3}	0.33512×10^{-4}
HP_2 $\longrightarrow$	0.53434×10^{-3}	0.14086×10^{-3}	0.35914×10^{-4}
RT_0 $\longrightarrow$	0.95306×10^{-2}	0.47632×10^{-2}	0.23813×10^{-2}

Table 1. Absolute errors of the solution measured in the L^2-norm for the test-problem.

diffusion-reaction equation. More precisely we compare in different situations the Hermite elements with the method RT_0 itself, the mixed least-squares formulation with linear finite element interpolations of both the unknown and its flux and an explicit scheme to deal with convection-diffusion equations studied in [2]. In most cases the Hermite finite elements provide valuable and reliable alternatives to cope with different critical situations encountered in practical applications. Among them we consider abrupt changes of a permeability tensor $\mathcal{K}$ for flows in porous media, or yet very large gradients occurring in boundary layers in the case of convection-diffusion processes at a high Péclet number.
Just to illustrate the results to be presented, we show the evolution of the errors of the approximate solution of the above stationary diffusion equation obtained with the methods HRT_0, HP_2, and RT_0, for the following data: $\mathcal{K}$ is the identity tensor, Ω is the square $(-1,1)\times(-1,1)$ and $f(x_1,x_2) = 1-(x_1^2+x_2^2)/2$. The exact solution is given by $u(x_1,x_2) = (1-x_1^2)(1-x_2^2)/4$. We solved the problem numerically using uniform meshes generated by first subdividing the computational domain into M^2 equal squares and then each one of these squares into two triangles by means of their diagonals parallel to the line $x_1 = x_2$. In Table 1 we display the absolute errors for increasing values of M, of the approximate solution obtained both with the Hermite methods HP_2 and HRT_0 and with the RT_0 element, in the norm of $L^2(\Omega)$.

As one can observe, the approximations obtained with both Hermite methods converge quadratically in the L^2-norm as predicted by the convergence analysis, whereas the ones obtained with RT_0 converge linearly.

3. Zienkiewicz-type N-simplex for incompressible flow

First we introduce some notations that we use in this Section: $(\cdot,\cdot)$ is the standard inner product of $L^2(\Omega)$ in scalar, vector or tensor version, with associated norm $\|\cdot\|$, and $(\cdot,\cdot)_D$ is the standard inner product of $L^2(D)$ with associated norm $\|\cdot\|_D$, for any proper subset D of Ω. $\|\cdot\|_{r,D}$ is the standard norm of Sobolev space $H^r(D)$ for $r \in \Re$ and $|\cdot|_{m,D}$ represents the standard semi-norm of Sobolev space $H^m(D)$, for $m \in I\!N$, D being a subset of Ω. We drop the subscript D whenever D is Ω itself.

3.1. Finite element description

To begin with we specify the Hermite finite elements we use to represent the velocity field locally, that is in every N-simplex T of a mesh, $N = 2, 3$. Let S_i be the vertices of T, $i = 1, \cdots, N+1$, and G its barycenter. We denote by λ_i the barycentric coordinate of T associated with S_i and set $h_{ij} = length[S_iS_j]$.

In the case $N = 2$ the elements are nothing but the well-known Zienkiewicz triangle in its two versions referred to here as Z_1 and Z_2, that is, with either incomplete or complete cubics, as defined in [13]. Just for better guidance we recall below that the nine degrees of freedom of Z_1 are the function values and the first order derivatives along the edges of T at its three vertices. For a convenient description of this Hermite finite element, the derivative along a given edge at a vertex belonging to it is always taken in the direction leading from this vertex to the other end of the edge under consideration. Denoting the bubble function of T by $\varphi = \lambda_1\lambda_2\lambda_3$ and $P_m(T)$ being the space of functions of degree less than or equal to m defined in T, the subspace of $P_3(T)$ associated with Z_1 is the one spanned by the set of nine linearly independent functions $[\{\zeta_i\}_{i=1}^3 \cup \{\zeta_{ij}\}_{i\neq j=1}^3]$, where $\zeta_i = \lambda_i^3 - \varphi$ and $\zeta_{ij} = \lambda_i^2\lambda_j + \varphi/2$. The nine canonical basis functions corresponding to the above specified degrees of freedom are given in [13]. In the case of Z_2 the above set of degrees of freedom is augmented with the function value at G.

The extension to tetrahedra of the Zienkiewicz triangle we consider in this work is described below:

Denoting by φ_{ijk} the function $\lambda_i\lambda_j\lambda_k$ the bubble function of face F_l of T, where the integers $i, j, k, l \in \{1, 2, 3, 4\}$ are assumed to be distinct, the analog of Z_1 still denoted this way is spanned by the set of sixteen linearly independent functions $[\{\zeta_i\}_{i=1}^4 \cup \{\zeta_{ij}\}_{i\neq j=1}^4]$, where $\zeta_i = \lambda_i^3 - \varphi_{ijk} - \varphi_{ijl} - \varphi_{ikl}$ and $\zeta_{ij} = \lambda_i^2\lambda_j + (\varphi_{ijk} + \varphi_{ijl})/2$. The set of sixteen degrees of freedom defining Z_1 in connection with the above basis are the function values at S_i and the first order derivatives at S_i along the three edges converging to this point, for $i = 1, 2, 3, 4$. Using the same notation for the analog of Z_2, this element is based on the space $P_3(T)$. The dimension of this space being twenty, the set of degrees of freedom defining Z_2 in connection with it are the function values at S_i and at the barycenter G_i of the face F_i opposite to S_i, together with the first order derivatives at S_i along the three edges converging to this point, for $i = 1, 2, 3, 4$.

The approximation properties of the above two- and three- dimensional elements were studied in [4] and [12] respectively. Let us briefly recall them.
First of all it is an easy matter to verify that the subspaces of $P_3(T)$ for elements Z_1 contain the space $P_2(T)$. Therefore if u is a function in $H^{l+2}(T)$, we can assert that its Z_l-interpolate in T $\pi_T^l(u)$ satisfies for suitable constants C_m^l independent of T and u,

$$\| u - \pi_T^l(u) \|_{m,T} \leq C_m^l h^{l+2-m} |u|_{l+2,T} \quad m = 0, 1, \ldots, l+2.$$

3.2. Solution method

For the sake of simplicity we assume that Ω is a polygon if $N = 2$ and a polyhedron if $N = 3$. Let $\mathcal{P}$ be a quasi-uniform family of partitions $\mathcal{T}_h$ of Ω into triangles or tetrahedra, satisfying the usual compatibility conditions for finite element meshes. Let h denote the maximum edge length of the elements in $\mathcal{T}_h$.

In all the sequel the letter C combined or not with other symbols represents constants independent of h. Also throughout this work $\mathbf{g}_h$ stands for piecewise cubic Hermite interpolates of $\mathbf{g}$ on $\partial\Omega$, assumed henceforth to belong to $\mathbf{H}^{5/2}(\partial\Omega)$. More specifically if $N = 2$ we mean the classical cubic interpolate at the vertices of $\mathcal{T}_h$ belonging to $\partial\Omega$, continuously differentiable along every straight portion of $\partial\Omega$, provided $\mathbf{g} \in \mathbf{H}^{5/2}(\partial\Omega)$. If $N = 3$ we define $\mathbf{g}_h$ to be the cubic Hermite interpolate of $\mathbf{g}$ on every face contained in $\partial\Omega$ of a tetrahedron of $\mathcal{T}_h$, using the degrees of freedom of the Zienkiewicz triangle, either with complete or incomplete cubics, according to the method being studied. Assuming that $\mathbf{f} \in \mathbf{H}^{l+1}(T)$ in every $T \in \mathcal{T}_h$, for l equal to 1 or 2, we will also work with approximations $\mathbf{f}_h^l$ of $\mathbf{f}$ in every element of $\mathcal{T}_h$, satisfying $\| \mathbf{f} - \mathbf{f}_h^l \|_T \leq C_l h^{l+1} |\mathbf{f}|_{l+1,T} \; \forall T \in \mathcal{T}_h$.

The problems to approximate (2) considered in this work are of the same kind as those proposed by Hughes-Franca-Balestra [7] and Douglas-Wang [5]. However in contrast to those works, we employ different mathematical tools in our convergence analysis, which simplify it significantly.

The solution methods to be studied use the pressure space Q_h^l, for $l = 1, 2$ respectively, defined as follows:

$$Q_h^l := \{ q \;/\; q \in C^0(\bar{\Omega}) \cap L_0^2(\Omega),\; q_{/T} \in P_l(T) \; \forall T \in \mathcal{T}_h \}$$

We associate with Q_h^l spaces $\mathbf{V}_h^l := [V_h^l]^2$ for $l = 1, 2$ to represent the velocity, both being constructed upon the plate Zienkiewicz element Z_l for $N = 2$ [13] or with its three-dimensional version defined above. This means that V_h^l is a space of continuous functions of degree less than or equal to three in each element of $\mathcal{T}_h$, whose gradient is also continuous at their vertices. While on the one hand V_h^2 consists of piecewise complete cubics in every N-simplex of $\mathcal{T}_h$, on the other hand in every element T a function of V_h^1 is spanned by all cubic functions, but the bubble functions, either of T itself if it is triangle or of the faces of T if it is a tetrahedron.

Let us denote by V_{h0}^l the space $V_h^l \cap H_0^1(\Omega)$, and introduce a broken $L^2(\Omega)$-inner product denoted by $(\cdot,\cdot)_h$, with associated norm $\| \cdot \|_h$, defined as follows for functions u and v defined only in the interior of the elements of $\mathcal{T}_h$:

$$(u, v)_h = \sum_{T \in \mathcal{T}_h} (u, v)_T; \qquad \| v \|_h = \sqrt{(v, v)_h}.$$

Now given a numerical parameter $\delta > 0$ to be specified later on, we consider the following problems to approximate (2), where l equals 1 or 2:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}_h^l \in \mathbf{V}_h^l \text{ and } p_h^l \in Q_h^l \text{ such that } \forall \mathbf{v} \in \mathbf{V}_{h0}^l \text{ and } \forall q \in Q_h^l \\ \delta(\mu \Delta \mathbf{u}_h^l - \nabla\, p_h^l, \mu \Delta \mathbf{v} - \nabla\, q)_h + \mu(\nabla\, \mathbf{u}_h^l, \nabla\, \mathbf{v}) \\ -(p_h^l, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u}_h^l, q) = -\delta(\mathbf{f}_h^l, \mu \Delta \mathbf{v} - \nabla q)_h + (\mathbf{f}_h^l, \mathbf{v}) \\ \mathbf{u}_h^l = \mathbf{g}_h \text{ on } \partial\Omega. \end{array} \right. \qquad (3)$$

3.3. A priori error estimates

We have proved in [3] the following a priori error estimate for problem (3):

Assume that $\mathbf{f} \in \mathbf{H}^{l+1}(\Omega)$, $\mathbf{u} \in \mathbf{H}^{l+2}(\Omega)$, $p \in H^{l+1}(\Omega)$ and $\mathbf{g}_{/\Gamma_i} \in \mathbf{H}^{l+2}(\Gamma_i)$ for $i = 1, 2, \ldots, m$, where the Γ_i's are m disjoint straight edges or plane faces whose union is $\partial\Omega$. If $\mathbf{f}_h^l$ is chosen such that $\| \mathbf{f} - \mathbf{f}_h^l \| \leq C_N |\mathbf{f}|_{l+1}$, and we take $\delta = h^2$, then there exist constants C_l such that the approximate solution of (2) obtained by solving problem (3) satisfy:

$$\left\{ \begin{array}{l} \| \nabla(\mathbf{u} - \mathbf{u}_h^l) \| + h \| \Delta(\mathbf{u} - \mathbf{u}_h^l) \|_h + \| p - p_h^l \| + h \| \nabla(p - p_h^l) \| \\ \leq C_l h^{l+1} \left[|\mathbf{f}|_{l+1} + |\mathbf{u}|_{l+2} + |p|_{l+1} + \left(\sum_{i=1}^{m} \| \mathbf{g} \|_{l+2,\Gamma_i}^2 \right)^{1/2} \right]. \end{array} \right. \quad (4)$$

3.4. Miscelaneous remarks

It seems important to stress some merits of the Hermite elements studied in this paper. First of all we can state that they have an a priori advantage over Lagrange elements of the same order in terms of cost. More specifically, we can compare our second order method with the Taylor-Hood element [6] based on a classical Lagrange quadratic representation of the velocity and a continuous piecewise linear representation of the pressure, which is also a second order element in the $H^1 \times L^2$-norm. A simple count shows that in the two-dimensional case, on the same mesh, the ratio between the number of velocity degrees of freedom of our second order element and the one of the Taylor-Hood element is roughly $3/4$. However in the three-dimensional case this ratio becomes even more favorable for it is reduced to about one half. Concerning our third order elements a fair comparison is to be made with the Lagrange cubics for the velocity and quadratics for the pressure. In this case the above specified ratios are ca. 0.6 in both two- and three-dimensions. Notice however that in the two-dimensional case we may use inner node static condensation for both third order elements being compared, and in this case the velocity degree of freedom ratio is reduced to a little more than 0.4.

Another feature of Hermite pseudo-C^1 elements like those we studied here, is the fact that second order derivatives can be computed element by element within acceptable accuracy, directly from the numerical velocity field. This can be achieved by first interpolating the velocity gradients continuously at the vertices of the mesh, using continuous piecewise linear functions, which can be differentiated in each element.

The Zienkiewicz triangle has been used by the authors to simulate viscous incompressible flow problems. Computational evaluations comparing the performance of their approach with well-established methods of the same order such as Taylor-Hood's [6] will be shown. Moreover encouraging experiments indicated that, even for $\delta = 0$, the Zienkiewicz' representation of the velocity yields converging and good quality numerical results.

In view of this the authors intend to further exploit these Hermite methods in the near future, in the simulation of flow on curved manifolds, in which an accurate determination of velocity derivatives is a must.

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