Blucher Mechanical Engineering Proceedings
May 2014, vol. 1 , num. 1
www.proceedings.blucher.com.br/evento/10wccm

10th World Congress on Computational Mechanics
8-13 July 2012 • São Paulo • Brazil

DYNAMIC ANALYSIS OF BUILDINGS USING THE FINITE ELEMENT METHOD

C. A. T. Lucena[1], P. C. O. Queiroz[2], A. L. H. C. El Debs[2], A. V. Mendonça[1]

[1]*Dept. of Mechanical Engineering. University Federal of Paraiba, Brazil*
taurinolucena@gmail.com
mendonca@ct.ufpb.br
[2]*Dept. of Structural Engineering. School of Engineering at the University of São Paulo, Brazil*
queiroz@sc.usp.br
analucia@sc.usp.br

Abstract. This paper deals with finite element vibration analysis of buildings. Each component of building is discretized by its appropriated finite element, that is, bar and beam element for the frame sub-structure, plate finite element for the slabs. By applying compatibility and equilibrium conditions, all sub-structural interactions are incorporated into the system in order to produce a more refined structural analysis of buildings. Other issues for building vibration such as shear deformation, rotatory inertia, and plate-beam eccentricity are investigated as well. Numerical examples are presented and compared with results from commercial numerical packages widespread.

Keywords: *FEM, Buildings, Dynamic*

1. INTRODUCTION

The Finite Element Method (FEM) is a numerical technique based on continuum discretization so that the body is divided into finite number of small parts named elements and by expressing the unknown variable fields in terms of assumed approximating functions within each element. These functions are expressed in terms of discrete points named nodes. Clough & Wilson [1] give interesting historical details about first steps to establish of FEM solutions. For example, in 1960 the designation of finite element method was coined by Ray Willian Clough when a static analysis of stress plane problem [2] was modeled using that new born technique. Since then FEM solutions have been received many other contribution and applied to a variety engineering problems. This is the case for vibration analysis of frame and plate [3]-[6]. Additional details on vibration FEM solutions including more complex structures such as shells can be found elsewhere, for example [3]-[9].

A specific topic in building structural analysis has been studied by many researchers is associated with eccentric relative position beam-plate at each floor. One of many strategies has established to deal with eccentricity problem is the rigid offset approach. Harik et al. [14] presented an analytical solution, Mukhopadhya [15] proposed a finite difference solution, Harik et al. [14], Araújo [16], Sapountzakis et al. [17], Deb et al. [18]-[19], gave finite element solutions, Tanaka et al. built boundary element solutions.

In this paper the influence of effects such as shear deformation, rotatory inertia, and plate-beam eccentricity into building vibration responses are analyzed by house-made program called EDF and commercial packaged Ansys.

2. ASSEMBLY OF MASS AND STIFFNESS MATRICES

In this section the strategy of assembling of each elemental mass and stiffness matrices into global matrices of the building is briefly discussed. The building analysis can be split into major problems, namely space frame and plates (in bending and/or in tension), and their interaction effects, see Figure 1.

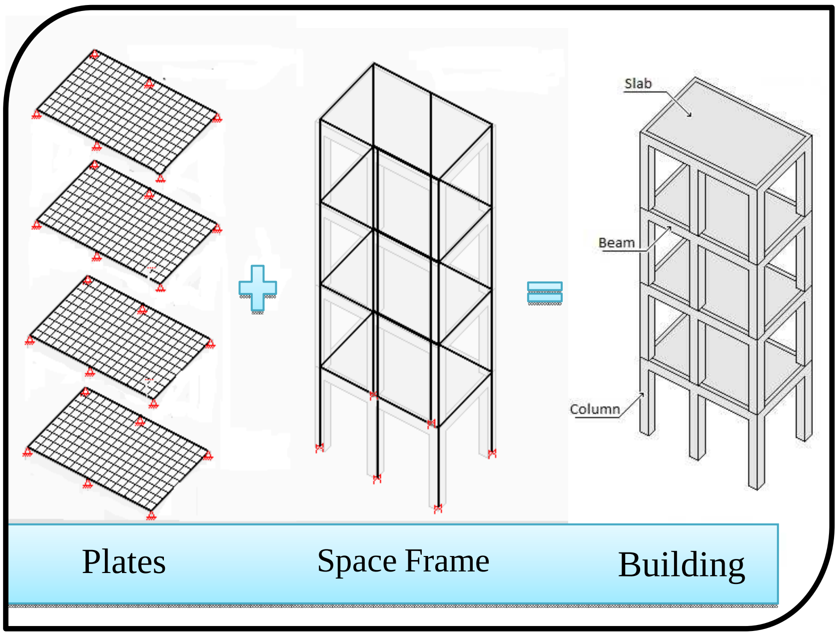


Figure 1. Building structural components.

In a space frame problem, the efforts in each member can be simultaneously mobilized due to axial, bending, and torsional actions. Both mass and stiffness matrices of a 3D frame member are well-known and they can be found in many works, such as Queiroz [9] and Lucena [10]. If only an isolated space frame is considered, the assembly of the global matrices can be done as shown in appendix - Figure 14(a). In this paper for the stretching and bending plate vibration problems were implemented respectively two elements namely CST (Constant Stress Triangle) and DST (Discrete Shear Triangle). Mathematical details for both elements can be found elsewhere, for example Petyt [12], Batoz and Lardeur [13], Lucena [10]. A relevant topic is the study of influence of beam-plate eccentricity of each floor to the global response of the building, see Figure 2.

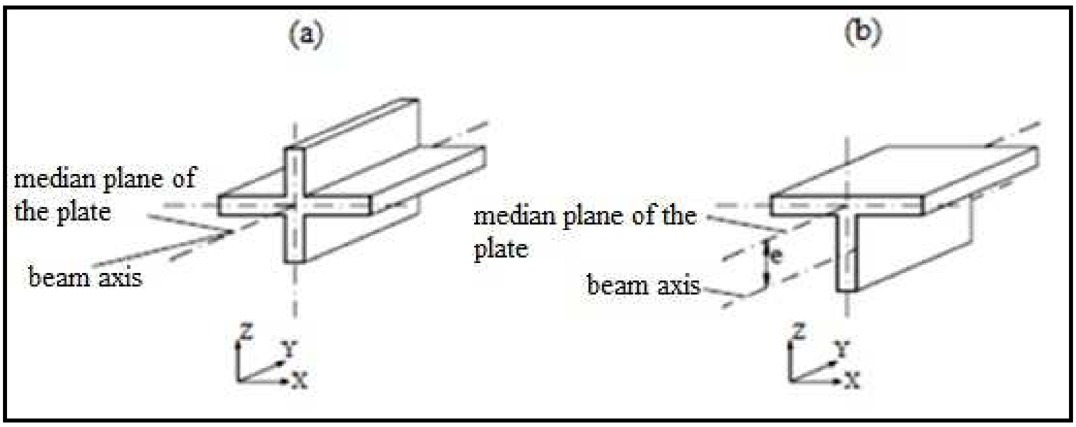


Figure 2. (a) Beam-plate configuration: (a) non-eccentric case; (b) eccentric case.

When a beam and a plate have no eccentricity implies nodal points of beam element coincide with some nodes of plate element, see Figure 2(a). In this case, no additional step is required and both beam and plate elemental contributions can be directly and independently assembled in global matrices of the structure. On other hand when beam-plate eccentricity exists (very usual situation in buildings, see Figure 2(b) it is necessary to reposition the degrees of freedom of the beam (usually located on its centroidal axis) onto a plane of the plate (usually median plane, see Figure 3) or vice-versa. Due to change of nodal location of beam, the energy conservation of system requires mass and stiffness matrices of the beam are transformed too.

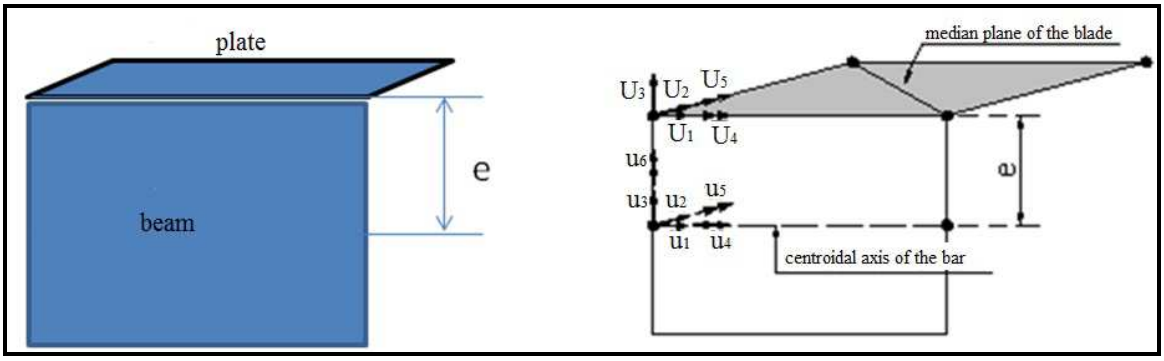


Figure 3. Local displacement vectors of the beam and their eccentricity.

Let there be {u} and {U} the global displacement vectors at beam centroidal axe and at plate median plane, respectively, and a relation can be written as follows.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = [\psi] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \text{ where } \qquad [\psi] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & e \\ 0 & 1 & 0 & 0 & -e & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

The final transformed matrices of the beam with respect to plate median plane system are:

$$\left[K^P\right] = [\psi]^T [K][\psi] \qquad (2)$$

$$\left[M^P\right] = [\psi]^T [M][\psi] \qquad (3)$$

In addition, the assembly of the transformed matrices given in Eq(2) and Eq(3) can be done as shown in appendix - Figure 14(b). For the building problem, an assembling strategy is to create three subsets of typical nodes. The first is associated with (stretching/bending) plate nodes that do not receive any contribution from space frame nodes, called “Plate nodes”. The second subset is called “Mixed nodes” and it receives both plate and space frame contributions”. The third subset called “Frame nodes” is formed by off-plate nodes. The subset dimensions are respectively given by $5N_L$, $6\,\mathrm{L_M}$ and $6\,\mathrm{L_F}$. Let there be a plate element with node orientation (i, j, k) and interacting with two (i,j) and (j,l) oriented frame elements, see Figure 4. Then, the assembly of building matrices from all elemental contribution can be done as shown in appendix - Figure 14(c).

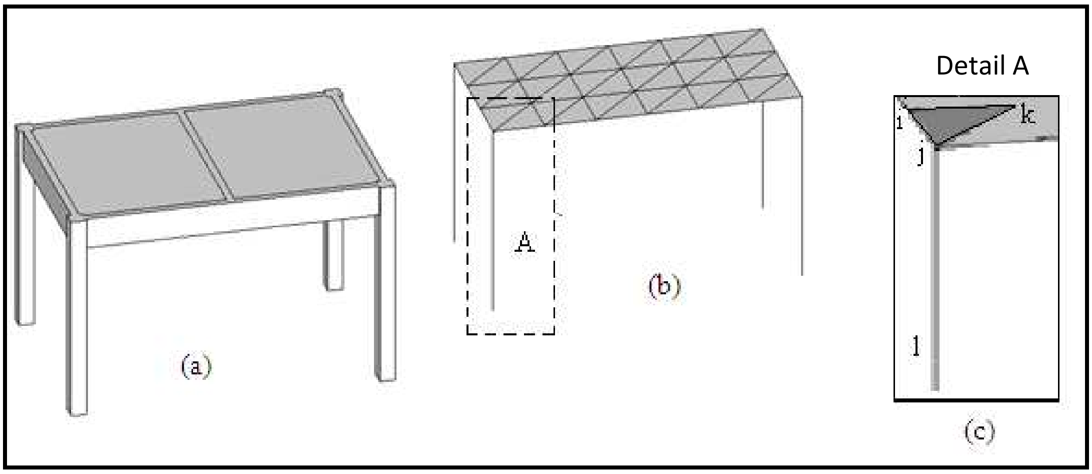


Figure 4. (a) Real structure; b) Discretized structure; c) Beam-Plate elements interacting at node j.

3. NUMERICAL RESULTS

All mathematical details of the finite element solution described in this paper were encoded in the program called EDF.FOR, see Lucena [11]. In order to check EDF.FOR performance, vibration analysis of two structures (space frame and four-storey building) is presented. The results of this house-made program are compared to commercial package ANSYS outputs.

3.1 Space Frame

This example was originally proposed by Petyt [12], see Figura 5. The mechanical and dimensions of the members are: Young modulus, $E = 219{,}9x10^9\, N/m^2$; density, $\rho = 7900 kg/m^3$ and length, $L = 1m$. Both Euler-Bernoulli and Timoshenko models are taken into account and the results from both programs are shown in Table 1.

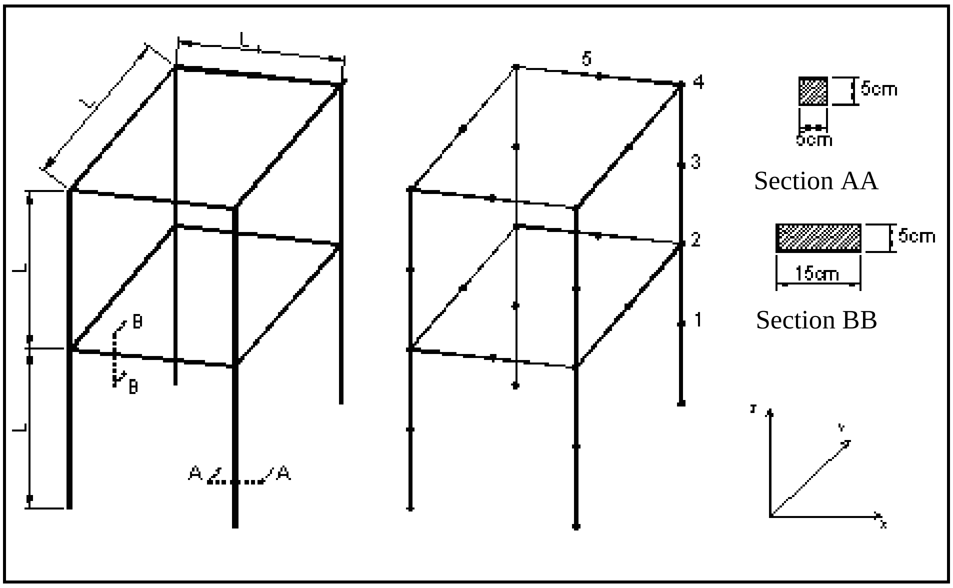


Figura 5. Space frame and its discretization.

Table 1. Frequency results (Hz)

Mode Number	EDF		Ansys	EDF		Ansys
	Model Euler[1]	Model Euler[2]	Model Euler	Model Timoshenko [1]	Model Timoshenko [2]	Model Timoshenko
1	11.808796	11.808643	11.808	11.76402563	11.76387476	11.774
2	11.808796	11.808643	11.808	11.76402563	11.76387476	11.808
3	15.4457303	15.411955	15.412	15.3966169	15.36295663	15.391
4	34.1154902	34.112377	34.111	33.98476808	33.98170942	33.992
5	34.1154902	34.112377	34.111	33.98476808	33.98170942	34.108
6	43.433024	43.330254	43.329	43.29169296	43.18938625	43.263
7	123.299665	123.18396	122.05	122.9237718	122.8097391	121.99
8	125.736015	125.6063	124.36	125.3376278	25.20986295	124.3
9	145.00127	144.88404	143.99	144.3209478	144.2060394	143.88
10	154.139833	153.98871	153.63	153.4583667	153.3105848	153.45
11	154.139833	153.98871	153.63	153.4583667	153.3105848	153.63
12	157.719628	157.54384	156.15	156.8369922	156.6655141	156.02
13	169.296732	169.14853	168.87	164.9313557	164.6636774	164.81
14	169.296732	69.148533	168.87	168.3485203	168.2042581	168.69
15	169.626978	169.32003	169.31	168.3485203	168.2042581	168.87
16	190.617268	190.16343	190.15	185.4286436	185.0307075	185.12
17	225.625976	225.32584	223.8	223.4873333	223.2009211	221.94
18	225.625976	225.32584	223.8	223.4873333	223.2009211	223.76
19	239.219681	238.87411	238.54	236.8261719	236.4969022	237.25
20	246.457127	246.0154	245.72	243.4084611	243.0062042	244.01

For sake of a better visualization, the results of Table 1 are plotted, see Figure 6.

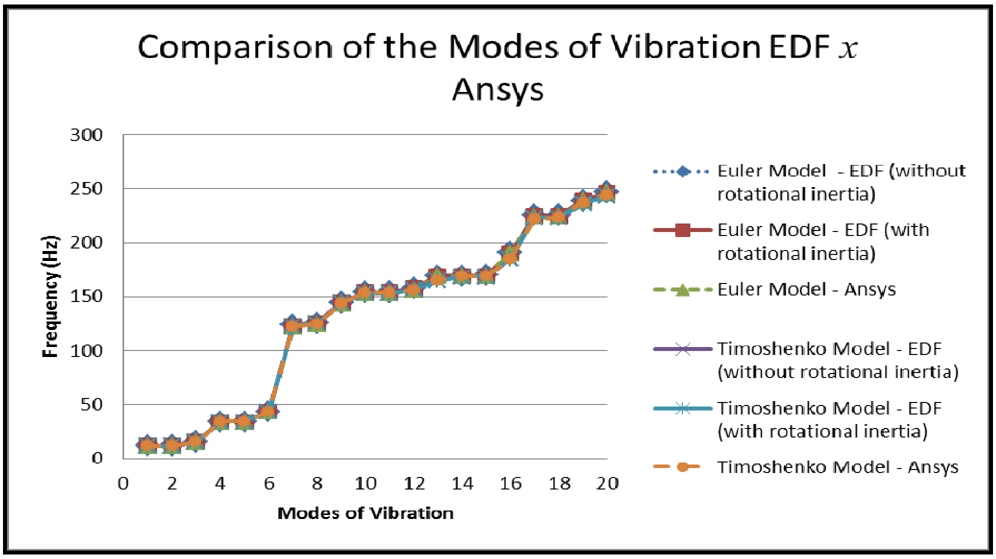


Figure 6. Space frame results.

---

[1] No rotatory inertia
[2] Rotatory inertia

3.2 Four-storey building

In this example vibration analysis for variety slender ratios of structural components of a four-storey building is done, see Figure 7. In addition the influence of plate-beam eccentricity, shear deformation and rotatory inertia effects in building vibration behavior is also considered. For all components, the following values for mechanical properties are set: Young modulus, $E = 21x10^9 N/m^2$; density, $\rho = 2500 kg/m^3$. For plate elements, Poisson´s ratio is $\upsilon = 0{,}30$ and all results are done using DKT element for bending plate contributions into the problem, except when is explicitly warned.

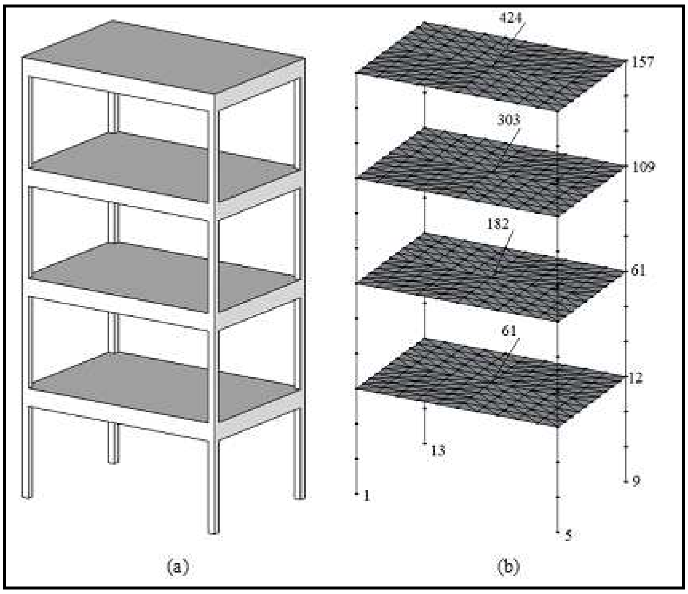


Figure 7. Four-storey building: (a) geometry, (b) discretization

The study was divided into three parts: firstly beam slender ratio influence in building vibration is analyzed under following assumptions. All plates have 10cm constant thickness and all columns have a typical $(20x20)cm$ square cross-section. A $(20xh)cm$ cross-section is set for all beams and two values are assumed for h.

Table 2 – Frequencies (Hz) for non-eccentric case: (plate h=0.10m) – (beam h=0.50m)

Mode	Euler-Bernoulli's beam		Timoshenko's beam	
Number	EDF	ANSYS	EDF	ANSYS
1	1.033634243568042	1.0335	1.026163178509705	1.0261
2	1.051321466365204	1.0512	1.042713789161397	1.0426
5	3.083254020001557	3.0829	3.059042393325857	3.0586
10	6.133236349035065	6.1322	6.090743238372382	6.0897
15	10.699237061042660	10.651	10.629603122588180	10.582
20	17.936212724087930	17.847	17.743458077855130	17.655
25	24.823920795071000	24.497	24.609889249841980	24.338
30	30.606705038030040	30.556	30.441299394962270	30.389
35	40.816869158106850	40.234	40.348166803980430	39.775
40	47.758536892544060	46.995	46.961864549614620	46.235
45	51.998620754883710	50.961	51.399784463171740	50.532

Table 3 –Frequencies (Hz) for eccentric case: (plate h=0.10m) – (beams h=0.50m)

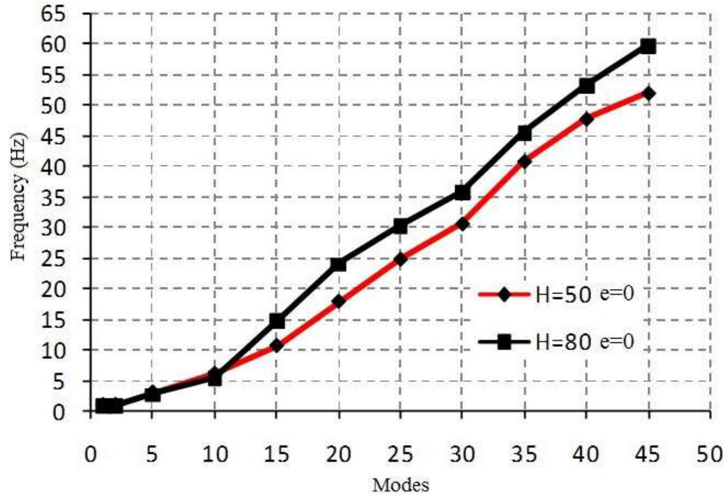
Mode Nº	Euler-Bernoulli's beam		Timoshenko's beam	
	EDF	ANSYS	EDF	ANSYS
1	1.061105210654078	1.0600	1.052878883863958	1.0518
2	1.069293282336665	1.0684	1.060060841738016	1.0593
5	3.124465230676489	3.1205	3.098809652840658	3.0951
10	6.143373253738758	6.1371	6.100320485187121	6.0943
15	12.818649259830260	12.742	12.705003588424920	12.630
20	20.492074748603760	20.333	20.289840881110260	20.189
25	27.490250113643460	27.043	27.221579028101290	26.782
30	31.587787232354530	31.561	31.405497422876140	31.376
35	43.533910399307850	42.826	43.291480405625190	42.625
40	50.194220128101630	48.459	50.045250822373090	48.315
45	53.158655994945020	51.930	52.177054714839100	51.022

Table 4 - Non-eccentric case: (plate h=0.10m) – (beam h=0.80m)

Mode Nº	Euler-Bernoulli's beam		Timoshenko's beam	
	EDF	ANSYS	EDF	ANSYS
1	0.9777737465142289	0.97767	0.9698353132263625	0.96973
2	0.9796925061525277	0.97959	0.9722366292403232	0.97213
5	2.845731851293956	2.8454	2.824630045531332	2.8243
10	5.497798210133395	5.4970	5.459845809899770	5.4591
15	14.831850434225120	14.708	14.716596862275180	14.596
20	24.099711540057980	23.971	23.685053731839450	23.524
25	30.405686660826370	29.792	30.182141949107190	29.584
30	35.744411088574980	34.730	35.716495690795860	34.703
35	45.569945054429910	44.477	44.918095831465320	43.615
40	53.179774314214070	52.360	52.789080492217070	51.107
45	59.693230687868560	58.063	58.688231895087200	58.148

Table 5 - Frequencies (Hz) for eccentric case: (plate h=0.10m) – (beam h=0.80m)

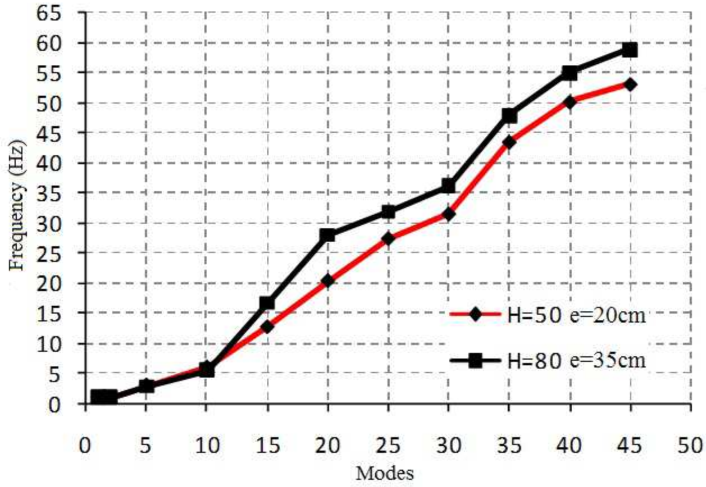
Mode Nº	Euler-Bernoulli's beam		Timoshenko's beam	
	EDF	ANSYS	EDF	ANSYS
1	0.9813890571155897	0.98293	0.9732779425289443	0.98125
2	0.9863427704428834	0.98709	0.9786756420889796	0.98592
5	2.861052757951518	2.8606	2.839452174069153	2.8580
10	5.500626179040095	5.4950	5.462595784379647	5.4940
15	16.706018819362260	16.555	16.558035888165120	16.411
20	28.044182252379500	27.728	27.343770282073450	27.046
25	31.864847655130900	31.222	31.620607422652910	30.992
30	36.210165576136050	35.291	35.955307263621120	34.968
35	47.867119090662290	44.218	47.636652351543720	44.003
40	55.012062684831130	52.910	54.902568106283380	51.712
45	58.919546022347830	55.373	58.042801175215840	54.735



Mode	Variation (%)
1	-5.4
2	-6.8
5	-7.7
10	-10.4
15	38.6
20	34.4
25	22.5
30	16.8
35	11.6
40	11.4
45	14.8

Figure 8. Beam ridigity influence on non-eccentric case

For sake a better visualization, the results of non-eccentric plate-beam case shown in Table 2 and Table 4 are plotted together in Figure 8. The eccentric plate-beam results in Table 3 and Table 5 are plotted in Figure 9.



Mode	Variation (%)
1	-7.5
2	-7.8
5	-8.4
10	-10.5
15	30.3
20	36.9
25	15.9
30	14.6
35	10.0
40	9.6
45	10.8

Figure 9. Beam ridigity influence on eccentric case

If the results of the thinner beams (h=50) in Figure 8 and in Figure 9 are taken as reference values, the first ten natural frequencies for thicker beams (h=80) decreased respectively between 5.4%and 10.4% for non-eccentric problem and 7.5% and 10.5% for eccentric case. On the other hand when range between 15th and 45th modes is checked, it can be seen that natural frequencies of the thicker beam problem are increased between 11.4% and 34.4% for the non-eccentric case and 9.6% to 36,9% for the eccentric problem. These major discrepancies are certainly due to more severe bending mode activation for thicker beams. A typical case is observed about 20th mode where mode configurations for both cases (h=50 and h=80) are shown in Figure 10.

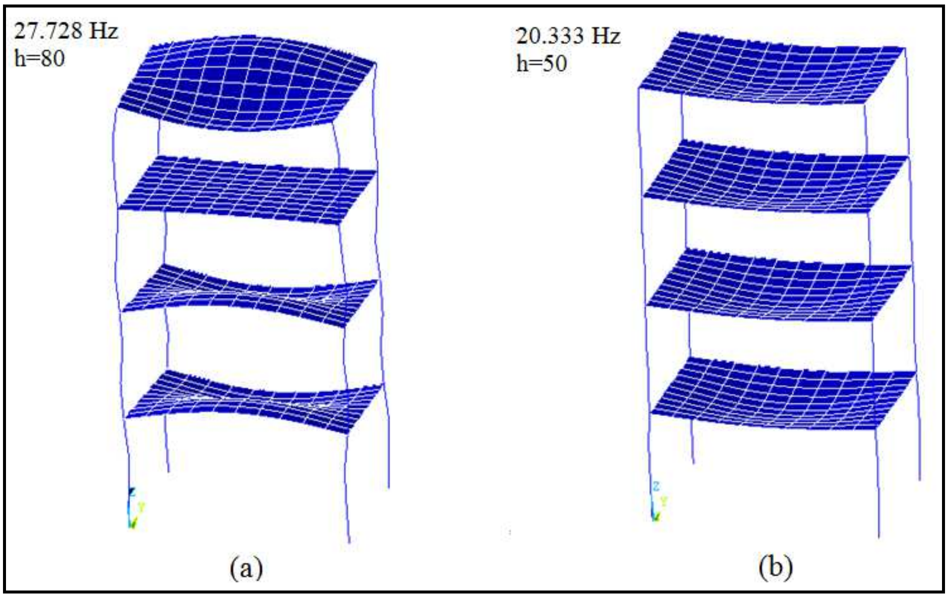


Figure 10. Settings of the 20th mode (a) h=80; (b) h=50

The second part of the building vibration analysis is concerned on plate slender ratio influence in building responses. Hence some assumption is done: beam-plate problem is always eccentric, all beams have the same cross-section $(20x80)cm$, and all columns cross-section is square with dimension 20cm.

The influence of plate thickness variation in the building vibration behavior is initially done and shear deformation effects are not included (classical plate model). Analyses for three thicknesses of the plate 10, 20 and 30 cm are independently done. By selecting DKT element, EDF and Ansys results are shown in Table 5, Table 6 and Table 7. For sake a better visualization the outputs are plotted together in Figure 11.

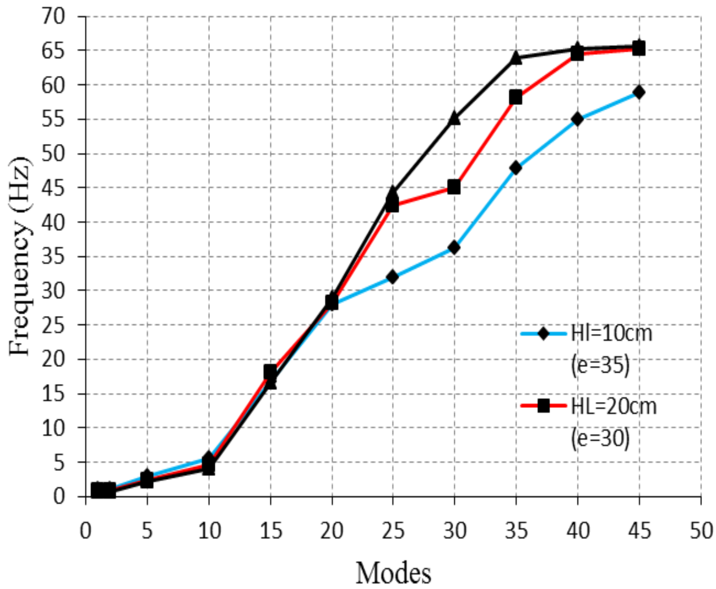
Table 6. Frequencies (Hz) eccentric case: (plate h=0.20m) – (beam h=0.80m) - column (0.20x0.20)m

Mode Nº	Euler-Bernoulli		Timoshenko	
	EDF	ANSYS	EDF	ANSYS
1	0.8293750409755191	0.83044	0.8227936436689241	0.82388
2	0.8340287604609957	0.83445	0.8276543806135503	0.82810
5	2.417162832657765	2.4167	2.399140468688497	2.3988
10	4.634355366607846	4.6318	4.602253750231750	4.5998
15	18.129607984803820	18.046	17.833315482366810	17.714
20	28.109150010122560	28.145	27.991753951863470	28.027
25	42.387049407920120	41.670	41.322183753097150	36.889
30	45.099012533007870	44.012	43.801345471503300	42.697
35	58.162430748004090	57.335	57.756874147415520	55.909
40	64.49996888117163	64.405	63.066791827934590	62.789
45	65.345883383739720	65.287	63.707594933563410	63.631

Table 7. Frequencies (Hz) eccentric case: (plate h=0.30m) – (beam h=0.80m) - column (0.20x0.20)m

Mode Nº	Euler-Bernoulli		Timoshenko	
	EDF	ANSYS	EDF	ANSYS
1	0.7314230031823287	0.73224	0.7259813796638660	0.72678
2	0.7357778526412748	0.73612	0.7303252140427594	0.73065
5	2.131274143423030	2.1311	2.115764573823893	2.1156
10	4.079733654343166	4.0784	4.051566408859258	4.0503

15	16.630121151899140	16.612	16.579815076595760	16.562
20	29.002845646654430	29.057	28.904550876622010	28.960
25	44.219052338206040	43.821	43.227278646436850	42.832
30	55.184507852580140	54.130	54.682751123006350	53.618
35	63.994146800298080	63.636	62.540515496259820	62.281
40	65.285823976308460	65.233	63.617427580988740	63.571
45	65.604209407136820	65.584	63.909799670753620	63.882



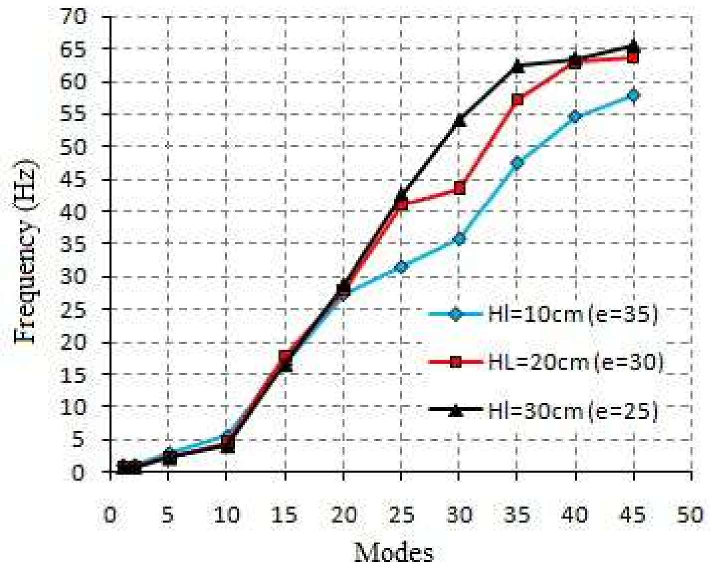
Mode	Variation (%) H_L=20	Variation (%) H_L=30
1	-15.5	-25.5
2	-15.4	-25.4
5	-15.5	-25.5
10	-15.7	-25.8
15	8.5	-0.5
20	0.2	3.4
25	33.0	38.8
30	24.5	52.4
35	21.5	33.7
40	17.2	18.7
45	10.9	11.3

Figure 11. Plate ridigity influence for eccentric case.

By assuming thinner plate results as reference values, it can be seen there is a decreasing of natural frequencies until 10th mode (15.7% for h=20 and 25.8% h= 30 cm). For higher modes (until 40th) the thicker plates increase natural frequencies. A complementary study of the earlier problem in order to check shear deformation effect in all building components is done as well. The vibration analysis is modeled by EDF (using both DST plate and Timoshenko beam elements) only and the results are shown in Table 8 and Figure 12.

Table 8. Frequencies (Hz) incorporating shear deformation effects for plate and beam

Mode Number	DST elements for plate and Timoshenko's beam (beam h=0.80m), column (0.20x0.20)m plate (h=10 cm) e=35cm	Plate (h=20 cm) e=30cm	Plate(h=30 cm) e=25cm
1	0.9732744677093985	0.8227266718092883	0.7257943847425563
2	0.9786749015925560	0.8276392615450652	0.7302702382631422
5	2.839449996129212	2.399105589554805	2.115640255869238
10	5.462592105083606	4.602237793494199	4.051521762546975
15	16.526669415551750	17.803260799809390	16.557976877659960
20	27.326817262917330	27.979698423999490	28.856040883028330
25	31.503036906434580	41.135618019665660	42.754293742999600
30	35.822998742607440	43.680109794382580	54.197368729408740
35	47.543719135812010	57.320421256622860	62.500908874808780
40	54.688519389612870	62.974795699346510	63.596349288916920
45	57.948194188817130	63.703264647440940	63.906124089558420



Mode	Variation (%)	
	H_L=20	H_L=30
1	-15.5	-25.4
2	-15.4	-25.4
5	-15.5	-25.5
10	-15.7	-25.8
15	7.7	-0.2
20	2.4	5.6
25	30.6	35.7
30	21.9	51.3
35	20.6	31.5
40	15.2	16.3
45	9.9	13.2

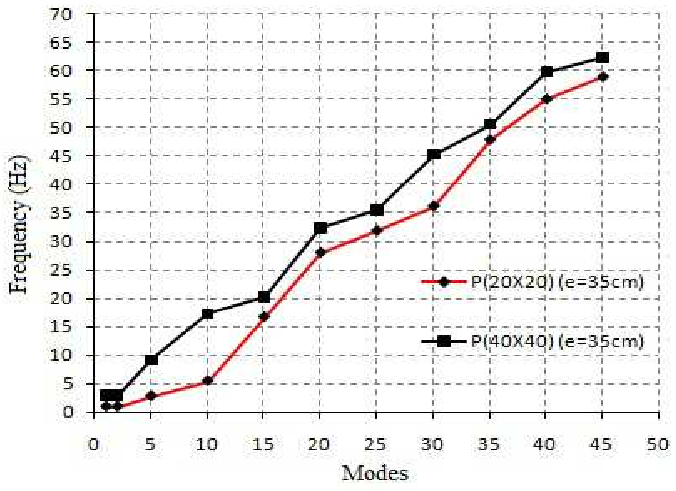
Figure 12. Plate rigidity influence and shear deformation effects on eccentric case

The results show there are no significative changes until 45th mode vibration responses when shear deformation was taken into account.

At last third part of the building vibration analysis in this paper is concerned about influence of variation of column cross-section dimensions. The assumptions for this analysis are: all plates has thickness 10cm and a cross-section $(20x80)cm$ is set for all beams. Two cross-section dimensions are set to the all columns, $(20x20)cm$ and $(40x40)cm$.. The EDF and Ansys results for $(40x40)cm$ cross-section columns in Table 9 are shown only. In Figure 13 both column cross-sections analyses are plotted.

Tabela 9. Frequencies (Hz) eccentric case: (plate h=0.10m) – (beam h=0.80m) - column (0.40x0.40)m

Mode Number	Euler-Bernoulli's beam		Timoshenko's beam	
	EDF	ANSYS	EDF	ANSYS
1	2.984353648437686	2.9881	2.897020117432950	2.9021
2	3.054102673539377	3.0591	2.942812889441475	2.9500
5	9.212390099187937	9.2083	8.899009872081662	8.9050
10	17.312097285493380	17.097	17.159695509391720	16.950
15	20.308793665539740	20.198	19.716451606859310	19.628
20	32.328691239677820	31.618	31.930651694976390	31.330
25	35.467487874409320	34.664	34.106090408379160	33.445
30	45.352876833737640	43.417	44.819779928222640	42.518
35	50.625448328801750	48.568	49.406304454311320	47.507
40	59.774830750619830	53.851	59.611505646993460	53.757
45	62.338234193366160	60.002	61.645643383805460	59.406



Mode	Variation (%)
1	204.1
2	209.6
5	222.0
10	214.7
15	21.6
20	15.3
25	11.3
30	25.2
35	5.8
40	8.7
45	5.8

Figure 13. Column rigidity influence on eccentric case

The results show lower modes are strongly affected when the column cross-section dimensions are changed.

4. CONCLUSIONS

The numerical results in the paper suggest results from EDF house-made program are in a good accordance with those run with commercial package Ansys. The changes in four-storey building vibration responses are analyzed, when slender ratios of each group of structural components are independently changed. The most severe changes for lower natural frequencies of the building is due to column cross-section variation. Shear deformation effects produce slight changes to lower natural frequencies of the building when compared to classical models.

Acknowledgments

The authors thank Capes and CNPq for financial support for this research.

5. REFERENCES

[1] Clough, R. W.; Wilson E. L. “Early Finite Element Research at Berkeley”, *Present at the Fifth U.S. National Conference on Computational Mechanics*, 1990.

[2] Clough, R. W., “The Finite Element Method in Plane Stress Analysis”, *Proc. 2nd ASCE Conf. On Eletronic Computation*, Pittsburg, Pa. Sept. 1960.

[3] Anderson, R.G., Irons, B.M., Zienkiewicz, O.C. Vibration and stability of plates using finite elements. Int. *Journal of Solids and Structures*, 4 1031-1005, 1968.

[4] Dickinson, S.M., Henshel, R.D. Clough-Tocher triangular plate-bending element in vibration, AIAA *Journal* 7 560-561, 1969.

[5] TOCHER, J. L. Analysis of Plate Bending Using Triangular Elements. *University of California, Berkeley*, Ph.D. Dissertation, 1962.

[6] Mackerle, J. Finite element vibration analysis of beams, plates and shells. *Shock and Vibration*, 6 97–109, 1969.

[7] Samanta, A. & Mukhopadhyay, M. Finite element static and dynamic analyses of folded plates. *Eng. Struct*., 21 277-871 1999.

[8] Lim, G.H. Vibration of plates and shells using finite elements (1996-1997). *Finite Elem. Anal. Design*, 31 223-30, 1999.

[9] Queiroz, P. C. O. Análise estática e dinâmica de estruturas reticuladas: ambiente de simulação em JAVA. *Universidade Federal da Paraíba*, Dissertação de mestrado 2010.

[10] Lucena, C. A. T. Análise estática e dinâmica de edifícios industriais utilizando-se o método dos elementos finitos. *Universidade Federal da Paraíba*, Dissertação de Mestrado, 2011.

[11] Viana, H. R. G. Análise estática e vibratória de placas utilizando-se o método dos elementos finitos. Universidade Federal da Paraíba, Dissertação de mestrado, 2008.

[12] Petyt, M. Introduction to finite element vibration analysis. *Cambridge University Press*, 1990.

[13] Batoz, J.L. & Lardeur, P.A. A discrete shear triangular nine dof element for the analysis of tick to very thin plates. *Int. J. Num. Meth. Eng*. V.28. p. 533-560, 1989.

[14] Harik, I. E. & Guo, M. Finite Element Analysis of Eccentrically Stiffened Plates in Free Vibration. *Computer and Structures*. 6, Vol. 49, pp. 1007-1015, 1993.

[15] Mukhopadhya, M. Stiffened Plates in Bending. *Computer and Structures*. 4, Vol. 50, pp. 541-548, 1994.

[16] Araújo, J. M. Avaliação dos métodos simplificados para cálculo de lajes maciças apoiadas em vigas flexíveis. *Teoria e Prática na Engenharia Civil*. Vol. 12, PP. 1-11, 2008.

[17] Sapountzakis, E. J. & Katsikadelis, J. T. Analysis of Plates Reinforced with Beams. *Computacional Mechanics*. Vol. 26, pp. 66-74, 2000.

[18] Deb, A. & Botton, M. Finite element models for Stiffened plates under transverse loading. *Computer and Structures*. 3, Vol. 28, pp. 362-372. 1988.

[19] Deb, A. & Deb, M. K. Analysis of Orthotropically Modeled Stiffened Plates. *International Journal of Solids Structures*. 5, Vol. 25, pp. 647-667, 1991.

[20] Tanaka, M. & Bercin, A. N. Static bending analysis of stiffened plates using the boundary element method. *Engineering Analysis with Boundary Elements*. Pp. 147-154, 1998.

APPENDIX

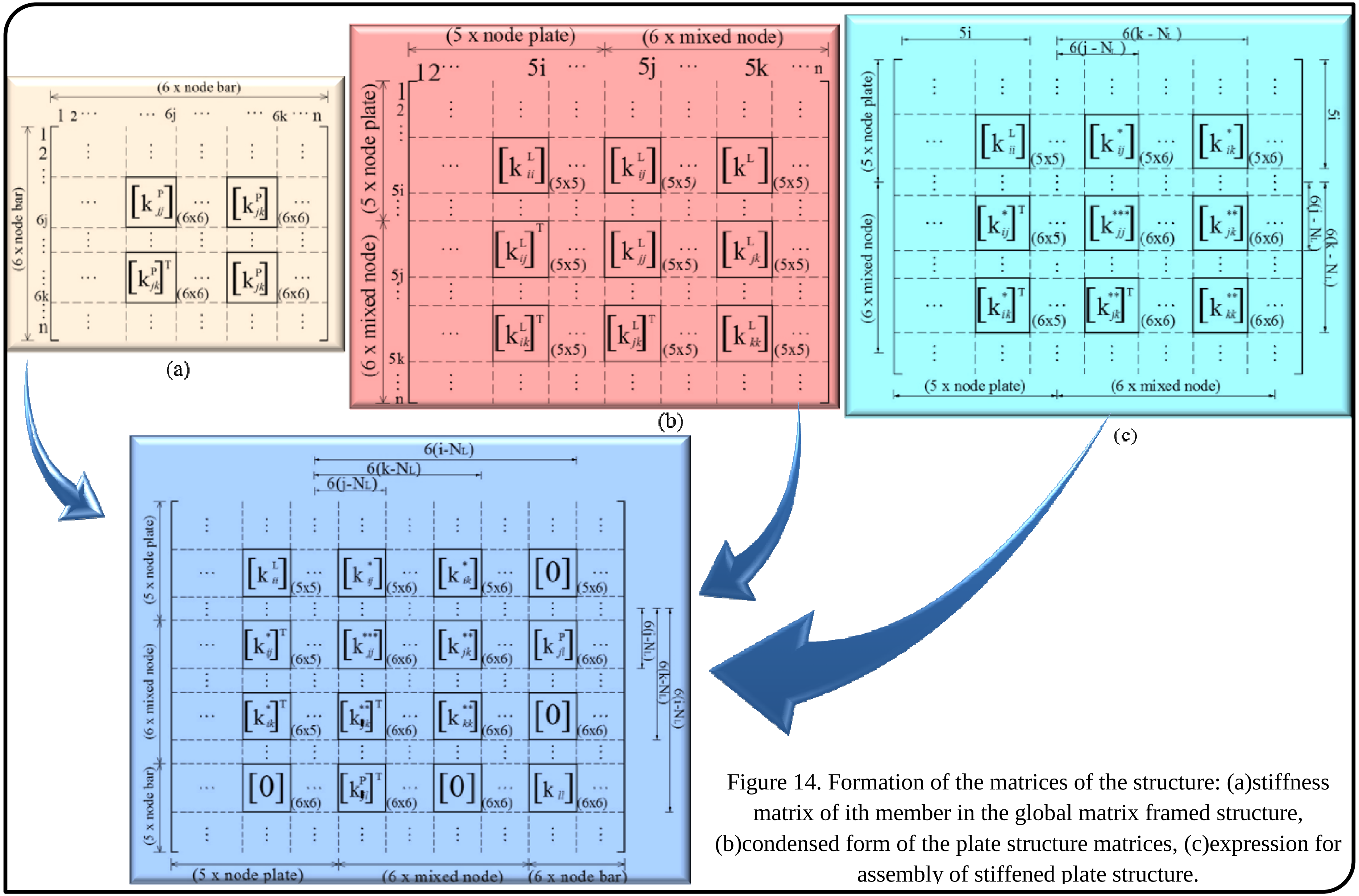


Figure 14. Formation of the matrices of the structure: (a)stiffness matrix of ith member in the global matrix framed structure, (b)condensed form of the plate structure matrices, (c)expression for assembly of stiffened plate structure.

The sub matrices in Figure 14, are given by:

$$\left[k_{ij}^{*}\right]_{5x6} = \left[\left[\mathrm{k}_{\mathrm{ij}}^{\mathrm{L}}\right]_{5\mathrm{x}6} \left[\,0\,\right]_{5\mathrm{x}1}\right] \tag{4}$$

$$\left[k_{ik}^{*}\right]_{5x6} = \left[\left[\mathrm{k}_{\mathrm{ik}}^{\mathrm{L}}\right]_{5\mathrm{x}5} \left[\,0\,\right]_{5\mathrm{x}1}\right]_{5x6} \tag{5}$$

$$\left[k_{ij}^{**}\right] = \begin{bmatrix} \left[k_{jj}^{L}\right]_{5x5} & \left[0\right]_{5x1} \\ \left[0\right]_{1x5} & 0 \end{bmatrix}_{6x6} + \left[\mathrm{k}_{\mathrm{jj}}^{\mathrm{P}}\right]_{6\mathrm{x}6} \tag{6}$$

$$\left[k_{jk}^{**}\right]_{6x6} = \begin{bmatrix} \left[k_{jk}^{L}\right]_{5x5} & \left[0\right]_{5x5} \\ \left[0\right]_{1x5} & \left[0\right] \end{bmatrix}_{6x6} + \left[\mathrm{k}_{\mathrm{jk}}^{\mathrm{P}}\right]_{6\mathrm{x}6} \tag{7}$$

$$\left[k_{kk}^{**}\right]_{6x6} = \begin{bmatrix} \left[k_{kk}^{L}\right]_{5x5} & \left[0\right]_{5x1} \\ \left[0\right]_{1x5} & \left[0\right] \end{bmatrix}_{5x6} + \left[\mathrm{k}_{\mathrm{kk}}^{\mathrm{P}}\right]_{6\mathrm{x}6} \tag{8}$$

$$\left[k_{jj}^{***}\right]_{6x6} = \begin{bmatrix} \left[k_{jj}^{L}\right]_{5x5} & \left[0\right]_{5x1} \\ \left[0\right]_{1x5} & \left[0\right] \end{bmatrix}_{6x6} + \overbrace{\left[\mathrm{k}_{\mathrm{jj}}^{\mathrm{P}}\right]_{6\mathrm{x}6}}^{Pilar} + \overbrace{\left[\mathrm{k}_{\mathrm{jj}}\right]_{6\mathrm{x}6}}^{barra} \tag{9}$$