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Mathematical aspects of transient solutions to the heat diffusion equation

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Abstract. *Problems involving the dynamics of heating or cooling of physical systems submitted to certain boundary conditions are of great importance in a wide range of situations. Since simple situations involving cooling of food where the temperature of each part of the system needs to be controlled and, in general, does not involve the presence of heat sources, even more complex situations like nuclear reactors, where there is a heat source which is dependent on position and the dependence of the reactor reactivity on the temperature requires a good knowledge of the dynamics of this quantity, both for safety and efficiency. Problems of the steady-state heat diffusion equation for a series of physical systems are easy to solve and can be found in a vast literature on the subject. To find the time evolution of temperature T (x, t) of a system submitted to a heat source, however, is not a simple task and this type of problem is usually treated in an approximate way. In this work, for one-dimensional and homogeneous system, equilibrium T(x) and transient T (x, t) solutions are presented for cases of the existence of sources permeating throughout the medium. Each system considered is supposed to have constant temperature in the edges. Calculations are presented for two cases: one is the case of a source non-dependent on position and time and the other is the case of a source with sinusoidal spatial dependence, a situation close to what would occur in a one-dimensional nuclear reactor. The extension to a three-dimensional case, homogeneous sphere with constant temperature on its surface will also be presented. The case involving the contact of two subsystems of different materials will also be analyzed.*

Keywords: heat diffusion equation, heating a sphere, transient temperature.

1. Introduction

Problems involving the dynamics of heating or cooling for physical systems submitted to certain boundary conditions are of great importance in a wide range of situations. Since simple situations involving cooling of food where the temperature of each part of the system needs to be controlled and, in general, does not involve the presence of heat sources, even more complex situations like nuclear reactors, where there is a heat source which is dependent on position and the dependence of the reactor reactivity on temperature requires a good knowledge of the dynamics of this quantity, both for safety and efficiency [1].

Problems of the steady-state heat diffusion equation for a series of physical systems are easy to solve and can be found in a vast literature on the subject. To find the time evolution of temperature T (x, t) of a system, submitted to a heat source, however, is not a simple task and this type of problem is usually treated in an approximate way [2], for example, using the Newton's law of cooling. The difficulty in dealing with such problems lies in the fact that, in general, transient solutions for diffusion equation are written as an infinite sum of eigenfunctions of the system, which, with appropriate coefficients, fall into stationary solutions when t tends to infinity. Such series, however, in many situations do not present at t equal zero, the well-defined derivative, which creates problems in terms of what happens, for example, with the heat flow within the system, which is proportional to the temperature gradient.

Once the appropriate series written in terms of a complete set of eigenfunctions of the system is found, which is compatible with the boundary and initial conditions of the problem, one can always construct the temperature derivative of a function by using the very definition of the derivative of a function and computational help; however, part of the analyticity of the solution is lost.

The problem of finding series whose derivative is not well defined is common in developments of Fourier series and there are strategies to overcome this condition [3]. This study demonstrates, with the aid of computations, the problem of the lack of a well-defined derivative at t equal zero, which does not necessarily exist for times different from zero, which makes the expressions that determine the temporal evolution of the system completely analytics, at least in the cases considered here. Thus, to analyze the results, computations are only required to make the sums of functions, a practice often indispensable when it comes to this type of solution.

In this work, for one-dimensional and homogeneous system, equilibrium T(x) and transient T (x, t) solutions are presented for cases of the existence of sources permeating throughout the medium. Each system considered is supposed to have constant temperature in the edges. Calculations are presented for two cases: one is the case of a source non-dependent on position and time and the other is the case of a source with sinusoidal spatial dependence, a situation close to what would occur in a one-dimensional nuclear reactor. The existence or not of a well-defined derivative for the series obtained will be discussed whenever necessary. The extension to a three-dimensional case, homogeneous sphere with constant temperature on its surface will also be presented. The case involving the contact of two subsystems of different materials will also be analyzed.

2 - Heat diffusion equation

The heat diffusion equation for a homogeneous system with conductivity k, heat capacity per unit of volume c and density ρ is given by

$$\nabla^2 T + \frac{q}{k} = \frac{1}{a^2}\frac{\partial T}{\partial t} \tag{1}$$

where a is the diffusion constant with a^2=(k/c) and q is the rate at which heat is created or absorbed per volume unit in the system. The heat flow is given by

$$J = -k\nabla T \tag{2}$$

2.1 - Linear heat flow

For a homogeneous system with unitary cross section exchanging heat with the environment only at its edges in positions x=0 and x=L, which we assume to have a constant temperature T_0, the heat diffusion is written as

$$\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{q}{k} = \frac{1}{a^2}\frac{\partial T(x,t)}{\partial t} \tag{3}$$

For $q(x,t) = q_0 \sin\left(\frac{\pi x}{L}\right)$, similar situation to that found in linear nuclear reactors in steady state diffusion regime. The steady state solution $T_{ss}(x,t)$ is easily obtained and results in

$$T_{ss}(x,t) = T_0 + \frac{q_0 L^2}{\pi^2 k}\sin\left(\frac{\pi x}{L}\right) \tag{4}$$

The t dependent solution $T(x,t)$ is written as $T(x,t) = T_{ss}(x) + T_{trans}(x,t)$ where the transient term is the solution to the diffusion equation without sources and, as usual $T_{trans}(x,t)$ is taken as a sum of eigenfunctions of the system

$$T_{trans}(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\omega_n t} \tag{5}$$

where $\omega_n = \frac{n^2\pi^2 a^2}{L^2}$, and coefficients A_n depends on the initial contitions.

For $T(x,0) = T_0$

$$T_0 + \frac{q_0 L^2}{\pi^2 k} \sin\left(\frac{\pi x}{L}\right) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = T_0 \tag{6}$$

and coefficients A_n results in

$$A_n = -\frac{q_0 L^2}{\pi^2 k} \delta_{1n} \tag{7}$$

and

$$T(x,t) = T_0 + \frac{q_0 L^2}{\pi^2 k} \sin\left(\frac{\pi x}{L}\right)\left(1 - e^{-\frac{\pi^2 a^2}{L^2} t}\right). \tag{8}$$

The heat flow per area unit is given by

$$J = -\frac{q_0 L}{\pi} \cos\left(\frac{\pi x}{L}\right)\left(1 - e^{-\frac{\pi^2 a^2}{L^2} t}\right). \tag{9}$$

The most interesting situation is to suppose the environment to have $T(t) = T_0 = cte$ and the temperature distribution of the system without sources to be $T(x,0) = T_2$ with $T_2 > T_0$, a cooling process. For this case $T_{ss}(x) = T_0$ and $T(x,t)$ is given by

$$T(x,t) = T_0 + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\omega_n t}. \tag{10}$$

Requiring $T(x,0) = T_2$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = T_2 - T_0 \tag{11}$$

after some simple calculations coefficients A_n are given by $A_n = -\frac{4}{n\pi}; n = odd$ and $A_n = 0; n = even$ and $T(x,t)$ results in

$$T(x,t) = T_0 + \sum_{n=odd} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-\omega_n t}. \tag{12}$$

The temperature distributions for $T_2 = 20°C$ and $T_0 = 10°C$ for several t values are presented in Figure 1, in this figure, the system parameters, in MKS units, are L=10, k=10, $a^2 = 0{,}1$, $q_0 = 20$. Note that the series for $T(x,t)$ do not have derivative for $t = 0$, despite this, for $t \neq 0$ the series

$$\frac{\partial T(x,t)}{\partial x} = \sum_{n=odd} \frac{4}{L} \cos\left(\frac{n\pi x}{L}\right) e^{-\omega_n t} \tag{13}$$

converge and the derivative of $T(x,t)$ obtained from the derivative definition of a function and some computational help coincides to the analytical expression (13). These curves are presented in figures 2 and 3 for two t values. Observe that for small t values more terms of the series (13) need to be taken into account to have a good agreement with the numerical calculations.

Temperature distribution $T(x,t)$

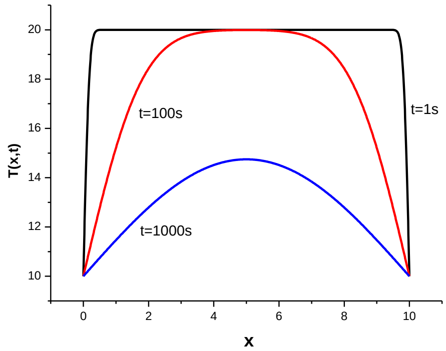

Figure 1. Temperature $T(x,t)$ calculated by using expression (12) for $t = 1s, 10s\ and\ 1000s$.

Derivatives of $T(x,t)$

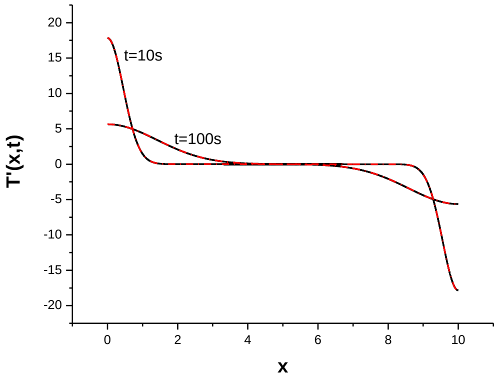

Figure 2. Derivatives of $T(x,t)$ calculated by using 50 terms of expression (13), black line, and numerical calculations, red line, for $t = 10s$ and t=100s.

Derivatives of $T(x,t)$

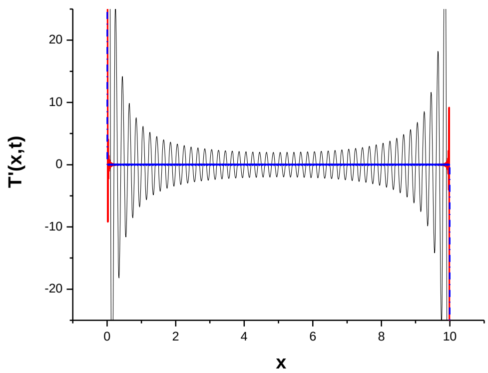

Figure 3. Derivatives of $T(x,t)$ calculated by using numerical calculations, blue line, and expression (13) with 50 terms, black line, 500 terms, red line, for $t = 0{,}0001s$.

2.2 The heating of a sphere.

Consider a homogeneous sphere of radius R, conductivity k, heat capacity per unit of volume c and density ρ, which we assume that the surface is kept at constant temperature T_0. If the source shows spherically symmetry, $q = q(r,t)$ and $T(\boldsymbol{r},0) = T(r,0)$, the heat diffusion equation (1) in spherical coordinates reduces to

$$\frac{2}{r}\frac{\partial T(r,t)}{\partial r} + \frac{\partial^2 T(r,t)}{\partial r^2} + \frac{q}{k} = \frac{1}{a^2}\frac{\partial T}{\partial t}. \tag{14}$$

For the case $q = q(r,t) = q_0$=constant, the steady state solution is given by

$$T_{ss}(r) = T_0 + \frac{q_0}{6k}(R^2 - r^2). \tag{15}$$

The maximum temperature occurs at $r = 0$ and results in

$$T_{max} = T_0 + \frac{Q}{8\pi k R}, \tag{16}$$

where Q is the total heat produced in the sphere per time unit.

The heat flux $J = -k\nabla T(r)$ results in

$$J = \frac{q_0}{3} r, \tag{17}$$

in complete agreement with the relation

$$J(R) * 4\pi R^2 = Q. \tag{18}$$

The solution $T(r,t)$ for a source acting on the system after $t = 0$ and with initial condition $T(r,0) = T_0$, is given by

$$T(r,t) = T_{ss}(r) + T_{trans}(r,t) \tag{19}$$

Where $T_{tras}(r,t)$ is written as

$$T_{trans}(r,t) = \sum_{n=1}^{\infty} A_n \frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} e^{-\omega_n t} \tag{20}$$

where $\omega_n = \frac{n^2\pi^2 a^2}{R^2}$. Requiring $T(r,0) = T_0$

$$T_0 + \frac{q_0}{6k}(R^2 - r^2) + T_{trans}(r,t) = T_0 \tag{21}$$

and

$$\sum_{n=1}^{\infty} A_n \frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} = \frac{q_0}{6k}(r^2 - R^2). \tag{22}$$

Coefficient A_n can be calculated by using the ortogonality of eigenfuctions of the system and results in

$$A_n = \frac{q_0}{6k}\frac{12R^3}{\pi^3}\frac{(-1)^n}{n^3}. \tag{23}$$

To verify that

$$\frac{12R^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} = (r^2 - R^2) \tag{24}$$

is an interesting exercise of mathematical physics and we show that on appendices A.

For the case which q presents a sinusoidal dependence on position

$$q = q_0 \frac{\sin\left(\frac{\pi r}{R}\right)}{\left(\frac{r}{R}\right)} \tag{25}$$

the steady state solution is given by

$$T_{ss}(r) = T_0 + \left(\frac{R}{\pi}\right)^2 \frac{q_0}{k} \frac{\sin\left(\frac{\pi r}{R}\right)}{\left(\frac{r}{R}\right)}. \tag{26}$$

The maximum temperature of the system results in

$$T_{max} = T_0 + \frac{Q}{4\pi k R} \tag{27}$$

where Q is the total heat produced in the sphere per time unit.

$$Q = 4\pi \int_0^R r^2 q_0 \frac{\sin\left(\frac{\pi r}{R}\right)}{\left(\frac{r}{R}\right)} dr \tag{28}$$

The solution $T(r,t)$ is written as

$$T(r,t) = T_{ss}(r) + T_{trans}(r,t) \tag{29}$$

where

$$T_{trans}(r,t) = T_{trans}(r,t) = \sum_{n=1}^{\infty} A_n \frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} \; e^{-\omega_n t}. \tag{30}$$

For a given initial condition $T(r,0) = T_0$ only the coefficient A_1 is different from zero and $T(r,t)$ is given by

$$T(r,t) = T_0 + \left(\frac{R}{\pi}\right)^2 \frac{q_0}{k} \frac{\sin\left(\frac{\pi r}{R}\right)}{\left(\frac{r}{R}\right)} \left(1 - e^{-\frac{\pi^2 a^2}{R^2} t}\right). \tag{31}$$

3. Analysis

Several interesting things may be emphasized by examining preceding calculations. First, it should be noted that empirical impositions, like the Newton's law of cooling, to solve the equations to cool down the process were not used at all. The solutions only depended on the initial and boundary conditions. Second, the problem of inexistence of a well-defined derivative to series like equation (12), for $t = 0$, as one can see in expression (13), does not exist for $t \neq 0$. The third question is that the mathematical solution $T(x,t)$, for equation (3), is given, in general, for expression (5) with any value of length L', and $\omega_{n'} = \frac{\mathrm{n}^2\pi^2\mathrm{a}^2}{\mathrm{L'}^2}$, the choice of a particular $L' = L$, the length of the system, is due to the necessity to keep the boundary condition $T(L,t) = T_0$. Different L' values do not respect this boundary condition. The similar apparent ambiguity occurs for the solution $T(r,t)$ in the three dimensional case. For the cases analyzed above, this is not really a problem since we must take $L' = L$ due to the boundary conditions. However, for heterogeneous systems like two coupled systems with different physical parameters and lengths L_1 and L_2, and the heat source restricted to act only on one of these materials, If one tries to write the solution $T(x,t)$ as the sum of solutions to the differential equation (3) for each region, it is necessary to keep $T(x,t)$ a continuous function in the interface between the two materials, so that this ambiguity cannot be solved without some additional consideration.

Appendices

Verifying the equality

$$\frac{12R^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} = (R^2 - r^2). \tag{A1}$$

Writing the left term as

$$-\frac{12R^3}{\pi^3}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^3}\frac{\sin\left(\frac{n\pi r}{R}\right)}{\mathrm{r}} \tag{A2}$$

and using

$$\sin\left(\frac{n\pi r}{R} + n\pi\right) = \sin\frac{n\pi r}{R}\cos n\pi \tag{A3}$$

denoting $\alpha = \pi\left(1 + \frac{r}{R}\right)$ we can write (A2) as

$$-\frac{12R^3}{\pi^3}\frac{1}{r}\sum_{n=1}^{\infty}\frac{\sin n\alpha}{n^3};\ \pi \le \alpha \le 2\pi. \tag{A4}$$

Using [4]

$$\sum_{n=1}^{\infty}\frac{\sin \mathrm{n}\alpha}{\mathrm{n}^3} = \frac{\pi^2\alpha}{6} - \frac{\alpha^2\pi}{4} + \frac{\alpha^3}{12} \tag{A3}$$

after some simple calculations we obtain

$$\sum_{n=1}^{\infty}\frac{\sin \mathrm{n}\alpha}{\mathrm{n}^3} = \frac{-\pi^3 \mathrm{r}}{12\mathrm{R}} + \frac{\pi^3\mathrm{r}^3}{12\mathrm{R}^3} \tag{A3}$$

and finally

$$-\frac{12R^3}{\pi^3}\frac{1}{r}\sum_{n=1}^{\infty}\frac{\sin \mathrm{n}\alpha}{\mathrm{n}^3} = -\frac{12R^3}{\pi^3}\frac{1}{r}\frac{\pi^3}{12}\left(\frac{\mathrm{r}^3}{\mathrm{R}^3} - \frac{\mathrm{r}}{\mathrm{R}}\right) = (\mathrm{r}^2 - \mathrm{R}^2). \tag{A4}$$

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