Structural Aspects of the Algebro-geometric Supermanifolds and Superdistributions

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Abstract

In this work, we analyse of the some structural aspects of algebro-geometric supermanifolds via Sheaf techniques and Algebraic Microlocal Analysis. The view contains a new construction of superdistribution and useful results on the wave-front set of such objects in algebro-geometric approach.

1 Introduction

Supergeometry is usually employed in the mathematical and theoretical physics in a rather heuristic way, and, accordingly, most expositions of that subject are heavily oriented towards physical applications. By way of contrast, in this work we wish to unfold a consistent and succinct investigation of the geometric objects, called supermanifolds, wich generalize differentiable manifolds by incorporating, in sense, anticommuting variables. So rather than turn our attention to physical issues, we will focus on the analysis of algebro-geometric foundations of the theory via Sheaf Theory.

2 Elements of Superalgebra

The main algebraic structures found in geometry are rings (of functions, differential forms, vector fields), modules (sheaves of modules), and homomorphisms of modules. Therefore, all basic laws of composition are naturally subdivided into multiplicative and additive law.

Let $\mathscr{A}$ a Grassmann algebra, such that $\mathscr{A}$ can naturally be decomposed as the direct sum $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$, where $\mathscr{A}_0$ consists of the even (commuting) elements and $\mathscr{A}_1$ consists of the odd (anti-commuting) elements in $\mathscr{A}$, respectively. Let χ_I denote the set of sequences $\{(\mu_1,\ldots,\mu_k)|1 \leq k \leq L; \mu_i \in \mathbb{N}; 1 \leq \mu_1 < \cdots < \mu_k \leq L\}$. A base of $\mathscr{A}$ is given by monomials of the form $\{\xi_\Omega, \xi^{\mu_1}\xi^{\mu_2},\ldots,\xi^{\mu_1}\xi^{\mu_2}\cdots\xi^{\mu_k}\}$ for all $\mu \in \chi_L$, such that $\chi_\Omega = \mathbb{I}$, where Ω represent the empty sequence in ξ_L, and $\xi^{(i)}\xi^{(j)} + \xi^{(j)}\xi^{(i)} = 0$ for $1 \leq i,j \leq L$.

The Grassmann algebra $\mathscr{A}_L$ is given the structure of a supercommutative algebra setting $\mathscr{A}_L = \mathscr{A}_{L,0} \oplus \mathscr{A}_{L,1}$ with $\mathscr{A}_{L,0}$ consisting of sums of combinations of even numbers of anticommuting generators, and $\mathscr{A}_{L,1}$of sums of combinations of odd numbers of anticommutating generators. An arbitrary element $p \in \mathscr{A}_L$ has the form

$$p = p_b + \sum_{(\mu_1,\ldots,\mu_k)} p_{\mu_1,\ldots,\mu_k}, \xi^{\mu_1} \ldots \xi^{\mu_k} \quad (1)$$

where p_b,$p_{\mu_1,\ldots,\mu_k}$ are real numbers. An even or odd element is specified by 2^{L-1} real parameters. For further details see [1].

3 Rings, Commutators, and Supercommutativity

Let $\mathscr{A}_L = \mathscr{A}_{L,0} \oplus \mathscr{A}_{L,1}$ be a $\mathbb{Z}_2$-graded ring. In the axiom of associativity $p(qr) = (pq)r$ there are no permutation of factors, and it is preserved in superalgebra in the previous form. The supercommutators of pair of elements p, $q \in \mathscr{A}_L$ is the element

$$[p,q] = pq - (-1)^{[p][q]} qp, \quad (2)$$

where $[p]$ is degree of the p. In expressions of this type it is always assumed the p, q are homogeneous, and the definition is extended to nonhomogeneous elements by additivity.

We say that p supercommutes with q if $[p,q] = 0$. The supercenter of $\mathscr{A}_L$ is $Z(\mathscr{A}_L) = \{p \in \mathscr{A}_L | \forall q \in \mathscr{A}_L,\ [p,q] = 0\}$. A ring morphism $f : \mathscr{A}_L \to \mathscr{B}$ defines on $\mathscr{B}$ the structure of an $\mathscr{A}_L$-algebra if $f(\mathscr{A}_L) \subset Z(\mathscr{B})$. All ring homomorphism preserve the gradation.

A ring $\mathscr{A}_L$ is supercommutative if $[p,q] = 0$ for all p,$q \in \mathscr{A}_L$. For further details see [1].

4 Topological Structure

The Grassmann algebra $\mathscr{A}_L$ equipped with the norm

$$||p||_q = \left(|p_b|^q + \sum_{(\mu)=1} |p_{\mu_1 \ldots \mu_k}|^q \right)^{1/q}, \quad (3)$$

becomes a Banach space. In fact $\mathscr{A}_L$ becomes a Banach algebra, i.e., $||\mathbb{I}|| = 1$ and $||pp'|| \leq ||p||||p'||$ for all p, $p' \in \mathscr{A}_L$.
A superspace must be constructed using as a building block a Grassmann-Banach algebra $\mathscr{A}_L$. Let $\mathscr{A}_L = \mathscr{A}_{L,0} \oplus \mathscr{A}_{L,1}$ be a Grassmann-Banach algebra. Then the (m,n)-dimensional superspace is the topological space $\mathscr{A}_L^{m,n} = \mathscr{A}_{L,0}^m \times \mathscr{A}_{L,1}^n$ which generalizes the space $\mathbb{R}^m$, consisting of the Cartesian product of m copies of the even part of $\mathscr{A}_L$ and n copies of the odd part. For further details see [2, 3].

5 Presheaves and sheaves

Let M be a topological space.
Definition: A *presheaf* $\mathscr{O}_M$ of supercommutative rings over M is a collection of algebras $\{\mathscr{O}(U) | U \text{ open in } M\}$ with the following properties.
(i) For a each pair U, V of open sets in M such that $V \subset U$ there exists a *restriction map* h_{UV} which is a ring homomorphisms from $\mathscr{O}(U)$ to $\mathscr{O}(V)$.
(ii) The restriction maps satisfy $h_{VW} \circ h_{UV} = h_{UW}$ whenever U, V and W are open in M and $W \subset V \subset U$.
The presheaf is sad to be a *sheaf* if in addition the following properties hold for every open cover $\{U_\alpha | \alpha \in \Gamma\}$ of each open set U in M.
(i) If f, g are in $\mathscr{O}(U)$, then $h_{UU_\alpha} f = h_{UU_\alpha} g$ for all $\alpha \in \Gamma$ implies that $f = g$.

(ii) If $f_\alpha \in \mathscr{O}(U_\alpha)$ is given for each $\alpha \in \Gamma$ with $h_{U_\alpha U_\alpha \cap U_\beta} f_\alpha = h_{U_\beta U_\alpha \cap U_\beta} f_\beta$ for all α, $\beta \in \Gamma$, then there exists $f \in \mathscr{O}(U)$ such that $f_\alpha = h_{UU_\alpha}$. For further details see [3, 4].

6 Algebro-geometric supermanifolds

General definition. A *smooth real algebro-geometric supermanifold* of dimensional (m,n) is defined to be a pair $(M, \mathscr{O}_M)$, consisting of a real m-dimensional manifold M and a sheaf of supercommutative rings $\mathscr{O}_M$ on it such that
(a) there exists an open cover $\{U_\alpha | \alpha \in \Gamma\}$ where for each α in Γ

$$\mathscr{O}(U_\alpha) \cong C^\infty(U_\alpha) \otimes \Gamma(\mathbb{R}^n), \qquad (4)$$

(b) if $\mathscr{N}$ is the sheaf of all nilpotents in the structure sheaf $\mathscr{O}$, then $(M, \mathscr{O}_M/\mathscr{N})$ is isomorphic to (M, C^∞).
In other words, a smooth real algebro-geometric supermanifolds is defined to be a locally decomposable superspace $(M, \mathscr{O}_M)$ such that $(M, \mathscr{O}_M/\mathscr{N})$ is isomorphic to an ordinary (purely even) manifold of the appropriate class, i.e., a Hausdorff space with countable basis and a sheaf of functions locally isomorphic to a domain in $\mathbb{R}^m$ with the real differentiable functions or a domain in $\mathbb{C}^m$ with the complex analytic functions.
Supermanifolds can be envisioned as superspaces with local systems of independent even-odd coordinates $(x^1, \ldots, x^m; \xi^1, \ldots \xi^n)$. As (ξ^j) we take a local basis of sections of the sheaf which realizes a local basis of sections of the sheaf which realizes a local decomposition. The decomposability implies that between x^j and ξ^j there are no other relations except consequences of supercommutativity. For further details see [3, 4, 5].

7 Distribution on a supermanifold $(M, \mathscr{O}_M)$

In our case, $M = \mathscr{A}_L^{m,0}$, so that introduced the concept of superdistributions such as the dual space of supersmooth functions in $\mathscr{A}_L^{m,0}$, with compact support, equipped with an appropriate topology, called *test functions*.
In order to define superdistributions, we need to give a suitable topological structure to the space $A_0^\infty(U, \mathscr{A}_L)$ of $\mathscr{A}_L$-valued superfunctions on an open set $U \subset \mathscr{A}_L^{m,0}$ which have compact support. Every A^∞ functions or superfunctions on a compact set $U \subset \mathscr{A}_L^{m,0}$ can be considered as a real-valued C^∞ function on $U \subset \mathbb{R}^{2^{L-1}(m)}$, regarding $\mathscr{A}_L^{m,0}$ and $\mathscr{A}_L$ as Banach spaces.
In fact, the identification of $\mathscr{A}_L^{m,0}$ with $\mathbb{R}^{2^{L-1}(m)}$ is possible because $(M = \mathscr{A}_L^{m,0}, \mathscr{O}_M/\mathscr{N})$ is isomorphic to the Banach manifold $\mathscr{A}_L^{m,0}$ and a sheaf of smooth functions on an m-dimensional manifold $\mathscr{A}_L^{m,0}$ locally isomorphic to a domain in $\mathbb{R}^{2^{L-1}(m)}$. In our words, $(M = \mathscr{A}_L^{m,0}, \mathscr{O}_M/\mathscr{N})$ is a C-functored space, where C be a category, and $\mathscr{O}_M/\mathscr{N}$ is a functor $C \to Sh(\mathscr{A}_L^{m,0})$. A superdistribution is a continuous linear functional $u : A_0^\infty(U) \to \mathscr{A}_L$, where $A_0^\infty(U)$ denotes the test superfunction space of $A^\infty(U)$ superfunctions with compact support in $K \subset U$. The continuity u on $A_0^\infty(U)$ is equivalent to its boundedness on a neighbourhood of zero, i.e., the set of numbers $u(\phi)$ is bounded for all $\phi \in A_0^\infty(U)$. The set all superdistributions in U is denoted by $\mathscr{D}'(U)$. For further details see [2, 6].

Proposition 1 [2]. A superdistribution u in $U \in \mathscr{A}_L^{m,n}$ is a continuous linear functional on $A_0^\infty(U)$ if and only if to every compact set $K \subset U$, there exists a constant C and $(m+n)$ such that

$$|u(\phi)| \leq C \sup_{\substack{|q|\leq m+n \\ z\in K}} |D^q(\phi)(z)|, \quad (5)$$

where $\phi \in G_0^\infty(K)$.

8 Wavefront set of a superdistribution

A superdistribution u on an open set $U \subset \mathscr{M}$, where $\mathscr{M}$ is a superspace, is sad to be infinitely differentiable in a conic subset Γ of cotangent bundle $T^*U \setminus \{0\}$ if it is infinitely differentiable in a neighbordhood of every point of Γ. The complement in $T^*U \setminus \{0\}$ of the union of all conic open sets in which u is infinitely differentiable is the **wave front set** of u and will be denoted by $WF(u)$. The $U \setminus \{0\} \subset \mathscr{M}_0$, where $\mathscr{M}_0$ is the body of superspace, excluding the trivial point $k_b = 0$.
A direction k_b for which Fourier transform of a superdistribution u shows to be of fast decrease is called to be regular direction of $\hat{u}$. Therefore, in order to determine whether (x_b, k_b) belongs to the wavefront set of u one must first to localize u around x_b, next to obtain Fourier transform $\hat{u}$ and finally to look at the decay in the direction k_b. Hence, the wavefront set not only describes the set of points where a superdistributions is singular, but it also localizes the frequencies that constitute these singularities.
Proposition 2 [2]. If $u, v \in \mathscr{D}'(U)$ satisfy

$$WF(u) \cap WF(v) = 0,$$

then the produt $u \cdot v$ is a well defined superdistribution in U.

For any open set $U \subseteq T^*U \setminus \{0\}$, we denote by c.sp.$(U)$ its conic span, i.e., the smallest conic set containing U, and we define

$$\mathscr{F}(U) := \frac{\mathscr{D}'(U)}{\{u \in \mathscr{D}'(U) : u \in C^\infty(\text{c.sp.}(U))\}}.$$

Its immediate to verify that the functor $U \to \mathscr{F}(U)$ gives a presheaf on $U \subseteq T^*U \setminus \{0\}$, and the associate sheaf is denoted *sheaf of supermicrodistributions*. Clearly every superdistribution $u \in \mathscr{D}'(U)$ defines a section of this sheaf whose support (in the sense of sheaf theory) is actually the wave-front set of u. For further details see [2, 5, 7].

9 Conclusions

We have analysed of the some structural aspects of algebro-geometric supermanifolds via Sheaf theoretic techniques and we have introduced a notion of superdistributions in superspaces exploring the categorical relations of such manifolds in an algebro-geometric approach. We have also obtained useful results on the singularity structure of superdistributions, here analysed in the context of the development of the mathematical tool of algebraic microlocal analysis and characterized in terms of the its wavefront set.

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