The Wheeler-DeWitt equation of f(G) gravity

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Abstract

We develop the quantization of a at cosmological model in the framework of the modified $f(G)$ theory of gravity, where G is the Gauss-Bonnet invariant. Therefore, a quantum model $f(G)$ is constructed and the corresponding WheelerDeWitt equation in minisuperspace is obtained, the central object of the canonical quantum cosmology.

1 Introduction

The General Relativity (GR) based on the Einstein-Hilbert action can not explain the acceleration of the Universe without taking into account the dark energy. Therefore, GR does not describe precisely gravity and it is quite reasonable to modify it in order to get theories that admit in ation and imitate the dark energy. The first tentative in this way is substituting Einstein-Hilbert term by an arbitrary function of the scalar curvature $\mathscr{R}$, this is the so-called $f(\mathscr{R})$ theory of gravity [1, 2]. In the same way, other alternative theory of modified gravity has been introduced, the so-called Gauss-Bonnet gravity, $f(G)$, as a general function of the Gauss-Bonnet invariant term G [3]. On the other hand, quantum cosmology may permit to determine the initial conditions of the Universe [4, 5]. A classical model depends on many initial data that must be fixed in order to have an agreement with observations. The quantum cosmology may lead to specific predictions concerning the values of at least some input parameters. In this sense, this work proposes a quantization of the $f(G)$ cosmological model. In next section, we give details on the construction of the classical model. In section 3, the Wheeler-DeWitt equation is obtained and final remarks made.

2 The classical model

Let us consider the following action for the $f(G)$ gravity

$$\mathscr{S} = \int d^4x\sqrt{-g}\left\{\frac{\mathscr{R}}{2\kappa} + f(G)\right\} + \mathscr{S}_m \ , \quad (1)$$

where $\kappa = 8\pi G_N$, with G_N is the gravitational Newtonian constant, $\mathscr{R}$ is the curvature scalar and G is the Gauss-Bonnet invariant , defined by $G = \mathscr{R}^2 - 4\mathscr{R}^{\mu\nu}\mathscr{R}_{\mu\nu} + \mathscr{R}^{\mu\nu\lambda\sigma}\mathscr{R}_{\mu\nu\lambda\sigma}$, $\mathscr{R}_{\mu\nu}$ and $\mathscr{R}_{\mu\nu\lambda\sigma}$ being the Ricci and Riemann tensors, respectively. In this work we will used the metric approach, according to which the action is varied with respect to the metric $g_{\mu\nu}$. Making this, one gets the following equations of motion

$$\begin{aligned} \mathscr{R}_{\mu\nu} & -\frac{1}{2}\mathscr{R}g_{\mu\nu} + 8\Big[\mathscr{R}_{\mu\lambda\nu\sigma} + \mathscr{R}_{\lambda\nu}g_{\sigma\mu} \\ - & \mathscr{R}_{\lambda\sigma}g_{\nu\mu} - \mathscr{R}_{\mu\nu}g_{\sigma\lambda} + \mathscr{R}_{\mu\sigma}g_{\nu\lambda} \\ + & \frac{\mathscr{R}}{2}\left(g_{\mu\nu}g_{\sigma\lambda} - g_{\mu\sigma}g_{\nu\lambda}\right)\Big]\nabla^\rho\nabla^\sigma f_G \\ + & (Gf_G - f)\,g_{\mu\nu} = \kappa T_{\mu\nu} \ . \end{aligned} \quad (2)$$

We are interested in $f(G)$ cosmological models and for simplicity, we consider the a flat FRW metric

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right) \ , \quad (3)$$

where, a is the scale factor. By making use of the line element (3), the equation of motion (2) is cast into two independent equations

$$\begin{aligned} 6H^2 &= 2\kappa\rho \\ &+ 24H^3\dot{G}f_{GG} - Gf_G + f \ , \end{aligned} \quad (4)$$

$$\begin{aligned} 4\dot{H} &+ 6H^2 = -2\kappa p + 8H^2\dot{G}^2 f_{GGG} \\ &+ 8H\left[H\ddot{G} + 2\left(\dot{H} + H^2\right)\dot{G}\right] f_{GG} \\ &- Gf_G + f \ , \end{aligned} \quad (5)$$

where $H = \dot{a}/a$ is the Hubble parameter, f_G, f_{GG} and f_{GGG}, the first, second and third derivative of the algebraic function f, respectively, and dot means derivative with respect to time t. It can be observed from (4)-(5) that the equations of motions are expressed in terms of the algebraic function f, its derivatives with respect to G, and H and G, and their time derivatives. Since, the Hubble parameter depends on the scale factor, one assume G, and a as the dynamical variables. To construct an effective Lagrangian, first of all, let us consider that, in general, the action is dependent of the sum of two functions $f_1(\mathscr{R})$ and $f_2(G)$. Then, the equations of FRW cosmology in $f(\mathscr{R},G) = f_1(\mathscr{R}) + f_2(G)$ gravity can be derived from a canonical point-like Lagrangian $\mathscr{L}\left(a,\dot{a},\mathscr{R},\dot{\mathscr{R}},G,\dot{G}\right)$. As done for spherical symmetry, Lagrange multipliers can be used to turn the expression of $\mathscr{R}$ and G and their derivatives into a constraint on the dynamics. A suitable choice of the Lagrange multipliers and simple integrations make the Lagrangian being canonical. It is important to put out here that we are working in a general case, where at the end of the calculations we will set $f_1(\mathscr{R}) = \mathscr{R}$, because of treating the modified $f(G)$ gravity. Spherically, the action for the general $f_1(\mathscr{R}) + f_2(G)$

is written as

$$\mathscr{S}ph = 2\pi^2 \int dta^3 \left\{ f_1(\mathscr{R}) \right.$$
$$- \lambda_1 \left[\mathscr{R} + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right]$$
$$\left. + f_2(G) - \lambda_2 \left[G - 24\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2} \right] \right\} . (6)$$

By assuming that this action does not suffer variation for small variation of $\mathscr{R}$ and G, one gets respectively the Lagrange multiplier $\lambda_1 = f_{1\mathscr{R}}$ and $\lambda_2 = f_{2G}$. Therefore, (6) becomes

$$\begin{aligned} \mathscr{S}ph &= 2\pi^2 \int dta^3 \left\{ f_1(\mathscr{R}) \right. \\ &- f_{1R} \left[\mathscr{R} + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \right] + f_2(G) \\ &\left. - f_{2G} \left[G - 24\frac{\ddot{a}}{a}\frac{\dot{a}^2}{a^2} \right] \right\} . \qquad (7) \end{aligned}$$

By integrating by parts, dropping out the term $\ddot{a}$, and remembering that $\mathscr{S}_{ph} \propto \int \mathscr{L} dt$, one gets

$$\begin{aligned} \mathscr{L} &= a^3 [f_1 - \mathscr{R} f_{1\mathscr{R}} + 6H^2 f_{1\mathscr{R}} \\ &+ 6H\dot{\mathscr{R}} f_{1\mathscr{R}\mathscr{R}} + f_2 \\ &- G f_{2G} - 8H^3 \dot{G} f_{2GG}] . \qquad (8) \end{aligned}$$

Recalling that we are working within $f(G)$ modified gravity, one have to set $f_1(\mathscr{R}) = \mathscr{R}$, such that $f_{1\mathscr{R}} = 1$ and $f_{1\mathscr{R}\mathscr{R}} = 0$. Moreover one may choose $f_2(G) = f(G)$ and $f_{2G} = f'$ and $f_{2GG} = f''$, prime being the derivative with respect to G. Thus, one gets the effective Lagrangian for $f(G)$ theory as

$$\begin{aligned} \mathscr{L}_{\text{eff}} &= \mathscr{L}_{\text{eff}}(a, \dot{a}, G, \dot{G}) \\ &= a^3 \Big[6H^2 + f - Gf' \\ &- 8H^3 \dot{G} f'' \Big] . \qquad (9) \end{aligned}$$

By making use of the effective Lagrangian (9), one gets the canonical momenta

$$p_a = \frac{\partial \mathscr{L}_{\text{eff}}}{\partial \dot{a}} = 12\dot{a}a - 24\dot{a}^2 \dot{G} f'' , \quad (10)$$

$$p_G = \frac{\partial \mathscr{L}_{\text{eff}}}{\partial \dot{G}} = -8\dot{a}^3 f'' , \qquad (11)$$

and the corresponding Hamiltonian reads

$$\begin{aligned} \mathscr{H} &= -\frac{3ap_G^{2/3}}{2f''^{2/3}} - \frac{p_a p_G^{1/3}}{2f''^{1/3}} \\ &- a^3 f + a^3 G f' . \qquad (12) \end{aligned}$$

3 The Wheeler-DeWitt equation and perspectives

In the Dirac-Wheeler-DeWitt canonical quantization of minisuperspace models one introduces a wave function which must fulfill the operator form of the constraint equation, that is,

$$\hat{H}\Psi = 0 , \qquad (13)$$

which plays the role of Schrödinger equation and is the heart of every canonical quantum gravity. Making the following replacement in the Hamiltonian

$$\begin{aligned} p_a \to \hat{p_a} &= -i\frac{\partial}{\partial a} \\ p_G \to \hat{p_G} &= -i\frac{\partial}{\partial G} , \qquad (14) \end{aligned}$$

we obtain the Hamiltonian operator and the Wheeler-DeWitt equation (Hamiltonian constraint) is written as

$$\begin{aligned} &\frac{2a\hbar^{2/3}}{2f''^{2/3}} \left(\frac{\partial}{\partial G} \right)^{2/3} \psi \\ &- \frac{\hbar^{4/3}}{2f''^{1/3}} \frac{\partial}{\partial a} \left[\left(\frac{\partial}{\partial G} \right)^{1/3} \psi \right] \\ &- \left(a^3 f - a^3 G f' \right) \psi = 0 . \qquad (15) \end{aligned}$$

The Wheeler-DeWitt equation in gravity $f(G)$ is a partial differential equation of fractional order [6, 7] for $\psi(a,G)$ in minisuperspace. Partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [8, 9, 10, 11]. Consequently, considerable attention has been given to the solutions of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. In the next work, we will seek solutions to the Wheeler-DeWitt equation (15) and we will adopt the Bohmian quantum theory to analyze quantum effects in primordial universe by deterministic trajectories for the scale factor.

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